Math 234, Practice Test #3

Show your work in all the problems.

1. Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane and lying outside the cylinder $x^2 + y^2 = 1$.

2. Evaluate the integral by changing to polar coordinates

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\, dx\, dy$$

3. Describe the region of integration. Convert the integral to spherical coordinates and evaluate it

$$\int_{-1}^{+1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dz\, dy\, dx$$

4. Sketch the region of integration, and write an integral with the order of integration reversed. Do not evaluate the integral.

$$\int_{0}^{4} \int_{-\sqrt{4-y}}^{(y-4)/2} dx\, dy$$

5. Find the centroid of the triangular region cut from the first quadrant by the line $x + y = 3$.

Solutions

1. The most convenient coordinates for this problem are cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$. We have to figure out when the paraboloid intersects the xy-plane. This is the case for $0 = z = 9 - x^2 - y^2$, i.e. $x^2 + y^2 = 9$ or $r = 3$. The integral is then given by

$$\begin{aligned}\int_0^{2\pi}\int_1^3\int_0^{9-r^2} r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_1^3 (9-r^2)r\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{9r^2}{2} - \frac{r^4}{4}\bigg|_1^3\right)d\theta \\ &= 32\pi\end{aligned}$$

2. The equations $x = \pm\sqrt{1-y^2}$ are equivalent to $x^2 + y^2 = 1$ which describes a circle with radius 1 centered at the origin. In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ this is given by $r = 1$. Hence

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy &= \int_0^{2\pi}\int_0^1 \ln(r^2+1)\,r\,dr\,d\theta \\ &\quad \text{(substitute } u = r^2+1,\ du = 2rdr) \\ &= \frac{1}{2}\int_0^{2\pi}\int_1^2 \ln(u)\,du \\ &= \frac{1}{2}\int_0^{2\pi}(u\ln(u) - u)\Big|_1^2 d\theta \\ &= \pi(2\ln(2) - 1)\end{aligned}$$

Recall from Calculus II that the integral of the logarithm function is calculated as follows:

$$\begin{aligned}\int \ln(x)dx &= \int 1\cdot\ln(x)dx \\ &\quad \text{(integration by parts } du = 1\,dx,\ v = \ln(x), \\ &\quad dv = dx/x,\ u = x) \\ &= x\ln(x) - \int dx \\ &= x\ln(x) - x + C\end{aligned}$$

3. The equation $z = \sqrt{x^2+y^2}$ describes a cone in upper half space $z \geq 0$ with tip at the origin and opening angle of $\pi/4$. The region of integration is bounded below by the cone and above by the horizontal plane

$z = 1$. In spherical coordinates the plane $z = 1$ corresponds to

$$1 = z = \rho \cos\phi \text{ i.e. } \rho = \sec\phi.$$

The converted integral is then

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta &= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \sec^3\phi \, \sin\phi \, d\phi \, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \sec\phi(\sec\phi\tan\phi) d\phi \, d\theta \\
&= \frac{2\pi}{3} \left. \frac{\sec^2\phi}{2} \right|_0^{\pi/4} \\
&= \frac{2\pi}{3}\left(1 - \frac{1}{2}\right) \\
&= \frac{\pi}{3}
\end{aligned}$$

4. The region of integration is the region enclosed by the parabola $y = 4 - x^2$ and the line $y = 4 + 2x$. Reversing the order of integration yields

$$\begin{aligned}
\int_{-2}^{0}\int_{4+2x}^{4-x^2} dy \, dx &= \int_{-2}^{0} (-x^2 - 2x) dx \\
&= -\left. \left(\frac{x^3}{3} + x^2\right) \right|_{-2}^{0} \\
&= \frac{4}{3}
\end{aligned}$$

5. The area of the triangle equals 9/2. We compute the first moments

$$\begin{aligned}
M_x &= \int_0^3 \int_0^{3-x} y \, dy \, dx \\
&= \frac{1}{2}\int_0^3 (3 - x)^2 dx \\
&= \frac{1}{2}\int_0^3 (9 - 6x + x^2) dx \\
&= \frac{1}{2}\left. \left(9x - 3x^2 + \frac{x^3}{3}\right) \right|_0^3 \\
&= \frac{9}{2}
\end{aligned}$$

and

$$
\begin{aligned}
M_y &= \int_0^3 \int_0^{3-x} x \, dy \, dx \\
&= \int_0^3 (3x - x^2) \, dx \\
&= \left(\frac{3x^2}{2} - \frac{x^3}{3} \right) \Bigg|_0^3 \\
&= \frac{9}{2}
\end{aligned}
$$

The center of mass is then given by

$$(\bar{x}, \bar{y}) = (M_y/M, M_x/M) = (1, 1)$$