MTH 234 Final Exam Review.

- Monday, December 12, 10:00am - 12:00 noon. (2 hours.)
- Be sure you know where your exam takes place.
- Coverage: Chapters 12-16.
  - Chapter 12, Sections 12.1-12.6.
  - Chapter 13, Sections 13.1-13.3.
  - Chapter 14, Sections 14.1-14.7.
  - Chapter 15, Sections 15.1-15.5, 15.7.
  - Chapter 16, Sections 16.1-16.8.
- From 10 to 20 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Plan for today:
- Extra review problems on Chapters 16, 15.
Plan for today:

- **Practice final exam April 30, 2001.**
- Extra review problems on Chapters 16, 15.

**Practice final exam April 30, 2001. Prbl. 1.**

Example

Given $A = (1, 2, 3)$, $B = (6, 5, 4)$ and $C = (8, 9, 7)$, find the following:

- $\vec{AB}$ and $\vec{AC}$.

  **Solution:**
  
  $\vec{AB} = \langle 6 - 1, 5 - 2, 4 - 3 \rangle$, hence $\vec{AB} = \langle 5, 3, 1 \rangle$. In the same way $\vec{AC} = \langle 7, 7, 4 \rangle$.

- $\vec{AB} + \vec{AC} = \langle 12, 10, 5 \rangle$.

- $\vec{AB} \cdot \vec{AC} = 35 + 21 + 4$.

- $\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 5 & 3 & 1 \\ 7 & 7 & 4 \end{vmatrix} = \langle (12 - 7), -(20 - 7), (35 - 21) \rangle$, hence $\vec{AB} \times \vec{AC} = \langle 5, -13, 14 \rangle$. 
Example

Find the parametric equation of the line through the point 
\( (1, 0, -1) \) and perpendicular to the plane 
\( 2x - 3y + 5x = 35 \). Then find the intersection of the line and the plane.

Solution: The normal vector to the plane \( \langle 2, -3, 5 \rangle \) is the tangent vector to the line. Therefore,
\[
\mathbf{r}(t) = \langle 1, 0, -1 \rangle + t \langle 2, -3, 5 \rangle,
\]
so the parametric equations of the line are
\[
\begin{align*}
x(t) &= 1 + 2t, \\
y(t) &= -3t, \\
z(t) &= -1 + 5t.
\end{align*}
\]
The intersection point has a \( t \) solution of
\[
2(1+2t) - 3(-3t) + 5(-1+5t) = 35 \quad \Rightarrow \quad 2+4t+9t-5+25t = 35
\]
\[
38t = 38 \quad \Rightarrow \quad t = 1 \quad \Rightarrow \quad \mathbf{r}(1) = \langle 3, -3, 4 \rangle.
\]

Example

The velocity of a particle is given by \( \mathbf{v}(t) = \langle t^2, (t^3 + 1) \rangle \), and the particle is at \( \langle 2, 1 \rangle \) for \( t = 0 \).

- Where is the particle at \( t = 2 \)?

Solution: \( \mathbf{r}(t) = \left\langle \left( \frac{t^3}{3} + r_x \right), \left( \frac{t^4}{4} + t + r_y \right) \right\rangle \). Since
\[
\mathbf{r}(0) = \langle 2, 1 \rangle,
\]
we get that \( \mathbf{r}(t) = \left\langle \left( \frac{t^3}{3} + 2 \right), \left( \frac{t^4}{4} + t + 1 \right) \right\rangle \).
Hence \( \mathbf{r}(2) = \langle 8/3 + 2, 7 \rangle \).

- Find an expression for the particle arc length for \( t \in [0, 2] \).

Solution: \( s(t) = \int_0^t \sqrt{\tau^4 + (\tau^3 + 1)^2} \ d\tau \).

- Find the particle acceleration.

Solution: \( \mathbf{a}(t) = \langle 2t, 3t^2 \rangle \).
Example

- Draw a rough sketch of the surface \( z = 2x^2 + 3y^2 + 5 \).

Solution: This is a paraboloid along the vertical direction, opens up, with vertex at \( z = 5 \) on the \( z \)-axis, and the \( x \)-radius is a bit longer than the \( y \)-radius.

- Find the equation of the tangent plane to the surface at the point \((1,1,10)\).

Solution: Introduce \( f(x, y) = 2x^2 + 3y^2 + 5, \) then

\[
L_{(1,1)}(x, y) = \partial_x f(1,1)(x-1) + \partial_y f(1,1)(y-1) + f(1,1).
\]

Since \( f(1,1) = 10, \) and \( \partial_x f = 4x, \) \( \partial_y f = 6y, \) then

\[
z = L_{(1,1)}(x, y) = 4(x-1) + 6(y-1) + 10.
\]

Example

Let \( w = f(x,y) \) and \( x = s^2 + t^2, \) \( y = st^2. \) If \( \partial_x f = x - y \) and \( \partial_y f = y - x, \) find \( \partial_s w \) and \( \partial_t w \) in terms of \( s \) and \( t. \)

Solution:

\[
\partial_s w = \partial_x f \partial_s x + \partial_y f \partial_s y = (x-y)2s + (y-x)t^2 = (x-y)(2s-t^2).
\]

Therefore, \( \partial_s w = (s^2 + t^2 - st^2)(2s - t^2). \)

\[
\partial_t w = \partial_x f \partial_t x + \partial_y f \partial_t y = (x-y)2t + (y-x)2st = (x-y)(2t - 2st).
\]

Therefore, \( \partial_t w = (s^2 + t^2 - st^2)2(1 - s). \)
Example
Find all critical points of the function \( f(x, y) = 2x^2 + 8xy + y^4 \) and determine whether they are local maximum, minimum, or saddle points.

Solution:
\[
\nabla f = \langle (4x + 8y), (8x + 4y^3) \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad \begin{cases} 
  x + 2y = 0, \\
  2x + y^3 = 0.
\end{cases}
\]

\[
-4y + y^3 = 0 \quad \Rightarrow \quad \begin{cases} 
  y = 0 \Rightarrow x = 0 \quad \Rightarrow \quad P_0 = (0, 0) \\
  y = \pm 2 \Rightarrow x = \mp 4 \quad \Rightarrow \quad P_1 = (4, -2), \\
  P_2 = (-4, 2)
\end{cases}
\]

Since \( f_{xx} = 4, f_{yy} = 12y^2, \) and \( f_{xy} = 8, \) we conclude that \( D = 3(16)y^2 - 4(16). \)
Example

Evaluate the integral \( I = \int_0^1 \int_x^{\sqrt{x}} y \, dy \, dx \) by reversing the order of integration.

Solution: The integration region is the set in the square \([0,1] \times [0,1]\) in between the curves \( y = x \) and \( y = \sqrt{x} \). Therefore,

\[
I = \int_0^1 \int_y^1 y \, dx \, dy = \int_0^1 y(y^2-y^4) \, dy = \int_0^1 (y^2-y^3) \, dy
\]

\[
I = \left. \left( \frac{y^3}{3} - \frac{y^4}{4} \right) \right|_0^1 = \frac{1}{3} - \frac{1}{4} \Rightarrow I = \frac{1}{12}.
\]

Example

Find the work done by the force \( \mathbf{F} = \langle yz, xz, -xy \rangle \) on a particle moving along the path \( \mathbf{r}(t) = \langle t^3, t^2, t \rangle \) for \( t \in [0,2] \).

Solution:

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt,
\]

where \( \mathbf{F}(t) = \langle t^3, t^4, -t^5 \rangle \) and \( \mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle \). Hence

\[
W = \int_0^2 \left( 3t^5 + 2t^5 - t^5 \right) \, dt = \int_0^2 4t^5 \, dt = \left. \frac{4}{6} t^6 \right|_0^2 = \frac{2}{3} 2^6.
\]

Therefore, \( W = 2^7 / 3 \).
Example
Show that the force field
\[ \mathbf{F} = \langle (y \cos(z) - yze^x), (x \cos(z) - ze^x), (-xy \sin(z) - ye^x) \rangle \]
is conservative. Then find its potential function. Then evaluate
\[ I = \int_C \mathbf{F} \cdot d\mathbf{r} \]
for \( \mathbf{r}(t) = \langle t, t^2, \pi t^3 \rangle \) for \( t \in [0, 1] \).

Solution: The field \( \mathbf{F} \) is conservative, since
\[ \partial_x F_y = \cos(z) - ze^x = \partial_y F_x, \]
\[ \partial_x F_z = -xy \sin(z) - ye^x = \partial_z F_x, \]
\[ \partial_y F_z = -x \sin(z) - e^x = \partial_z F_y. \]
The potential function is a scalar function \( f \) solution of
\[ \partial_x f = y \cos(z) - yze^x, \quad \partial_y f = x \cos(z) - ze^x, \quad \partial_z f = -xy \sin(z) - ye^x. \]

Example
Show that the force field
\[ \mathbf{F} = \langle (y \cos(z) - yze^x), (x \cos(z) - ze^x), (-xy \sin(z) - ye^x) \rangle \]
is conservative. Then find its potential function. Then evaluate
\[ I = \int_C \mathbf{F} \cdot d\mathbf{r} \text{ for } \mathbf{r}(t) = \langle t, t^2, \pi t^3 \rangle \text{ for } t \in [0, 1]. \]

Solution: Recall: \( f = xy \cos(z) - yze^x + h(z). \)
Introduce \( f \) into the equation \( \partial_z f = -xy \sin(z) - ye^x \), that is,
\[ -xy \sin(z) - e^x + h'(z) = -xy \sin(z) - ye^x \quad \Rightarrow \quad h'(z) = 0. \]
So, \( h(z) = c, \) a constant, hence \( f = xy \cos(z) - yze^x + c. \)
Finally \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,1,\pi)} df = f(1,1,\pi) - f(0,0,0). \)
So we conclude that \( \int_C \mathbf{F} \cdot d\mathbf{r} = -(1 + \pi e). \)


Example
Use the Green Theorem to evaluate the integral \( \int_C F_x \, dx + F_y \, dy \)
where \( F_x = y + e^x \) and \( F_y = 2x^2 + \cos(y) \) and \( C \) is the triangle with vertices \((0,0), (0,2)\) and \((1,1)\) traversed counterclockwise.

Solution: Denoting \( \mathbf{F} = \langle F_x, F_y \rangle \), Green’s Theorem says
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA = \iint_S (\partial_x F_y - \partial_y F_x) \, dA. \]
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (4x - 1) \, dx \, dy = \int_0^1 \int_y^{2-y} (4x - 1) \, dx \, dy. \]
A straightforward calculation gives \( \int_C \mathbf{F} \cdot d\mathbf{r} = 3. \)
Example
Find the surface area of the portion of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the plane \( z = 0 \). Use polar coordinates to evaluate the integral.

Solution:
\[
A(S) = \int \int \sigma, \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} dx dy
\]
where \( f = x^2 + y^2 + z - 4 \). Therefore,
\[
\nabla f = \langle 2x, 2y, 1 \rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{1 + 4x^2 + 4y^2}, \quad \nabla f \cdot k = 1.
\]
\[
A(S) = \int_{0}^{2\pi} \int_{0}^{2} \sqrt{1 + 4r^2} r dr d\theta, \quad u = 1 + 4r^2, \quad du = 8r dr.
\]
The finally obtain \( A(S) = (\pi/6)(173/2 - 1) \).

Example
Use the Stokes Theorem to evaluate
\[
I = \int \int S [\nabla \times (y \mathbf{i})] \cdot n \ d\sigma
\]
where \( S \) is the hemisphere \( x^2 + y^2 + z^2 = 1 \), with \( z \geq 0 \).

Solution: \( \mathbf{F} = \langle y, 0, 0 \rangle \). The border of the hemisphere is given by the circle \( x^2 + y^2 = 1 \), with \( z = 0 \). This circle can be parametrized for \( t \in [0, 2\pi] \) as
\[
\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle,
\]
and we also have \( \mathbf{F}(t) = \langle \sin(t), 0, 0 \rangle \). Therefore,
\[
\int \int S (\nabla \times \mathbf{F}) \cdot n \ d\sigma = \int_{0}^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t) \ dt = -\int_{0}^{2\pi} \sin^2(t) \ dt
\]
\[
\int \int S (\nabla \times \mathbf{F}) \cdot n \ d\sigma = -\frac{1}{2} \int_{0}^{2\pi} [1 - \cos(2t)] \ dt.
\]
Example

Use the Stokes Theorem to evaluate \( I = \iint_S \left( \nabla \times (yi) \right) \cdot n \, d\sigma \)

where \( S \) is the hemisphere \( x^2 + y^2 + z^2 = 1, \) with \( z \geq 0. \)

Solution: \( \iint_S (\nabla \times F) \cdot n \, d\sigma = -\frac{1}{2} \int_0^{2\pi} \left[ 1 - \cos(2t) \right] \, dt. \)

Recall that

\[
\int_0^{2\pi} \cos(2t) \, dt = \frac{1}{2} \left( \sin(2t) \right)_{0}^{2\pi} = 0.
\]

Therefore, we obtain

\[
\iint_S (\nabla \times F) \cdot n \, d\sigma = -\pi.
\]

MTH 234 Final Exam Review.

Plan for today:

- Extra review problems on Chapters 16, 15.
Example

A particle moves along a curve with velocity function \( v = i + \sqrt{2} t j + t^2 k \) for \( t \in [1, 2] \).

(a) If \( r(1) = \langle 0, \sqrt{2}/2, 2/3 \rangle \), find \( r(t) \).

\[
\begin{align*}
  r(t) &= (t + c_x)i + \left( \frac{\sqrt{2}}{2} t^2 + c_y \right)j + \left( \frac{t^3}{3} + c_z \right)k. \\
  r(1) &= \langle 1 + c_x, \frac{\sqrt{2}}{2} + c_y, \frac{1}{3} + c_z \rangle = \langle 0, \frac{\sqrt{2}}{2}, \frac{2}{3} \rangle \\
  c_x &= -1, \quad c_y = 0, \quad c_z = 1/3. \quad \text{(You finish.)}
\end{align*}
\]

(b) Distance traveled by the particle from \( t = 1 \) to \( t = 2 \).

Arc length:
\[
d = \int_1^2 |v(t)| \, dt = \int_1^2 \sqrt{1 + 2t^2 + t^4} \, dt
\]
\[
d = \int_1^2 \sqrt{(1 + t^2)^2} \, dt = \int_1^2 (1 + t^2) \, dt. \quad \text{(You finish.)}
\]

(c) Find the acceleration. \( a = \sqrt{2} j + 2t k \). \quad \triangleleft

Example

(a) Sketch the integration region of \( I = \int_1^e \int_0^{\ln(x)} y \, dy \, dx \).

The integration region is below \( y = \ln(x) \), above \( y = 0 \), for \( x \in [1, e] \).

(b) \( I = \int_0^1 \int_{e^y}^e y \, dx \, dy \).

(c) Evaluate \( I \).

\[
I = \int_0^1 \int_{e^y}^e y \, dx \, dy = \int_0^1 y(e - e^y) \, dy.
\]
\[
I = e \int_0^1 y \, dy - \int_0^1 ye^y \, dy.
\]

(Integrate by parts.) \quad \triangleleft
Plan for today:
- Extra review problems on Chapters 16, 15.

Remark on Chapter 16.

Remark: The normal (flux) form of Green’s Theorem is a two-dimensional restriction of the Divergence Theorem.

- The Divergence Theorem: \( \int_S F \cdot n \, d\sigma = \int_D (\nabla \cdot F) \, dv \).
- Normal form of Green’s Thrm: \( \oint_C F \cdot n \, ds = \int_S (\nabla \cdot F) \, dA \).

Remark: The tangential (circulation) form of Green’s Theorem is a particular case of the Stokes Theorem when \( C, S \) are flat \( (z = 0) \).

- The Stokes Theorem: \( \oint_C F \cdot dr = \int_S (\nabla \times F) \cdot n \, d\sigma \).
- Tang. form of Green’s Thrm: \( \oint_C F \cdot dr = \int_S (\nabla \times F) \cdot k \, dA \).
Chapter 16, Integration in vector fields.

Example

Use the Divergence Theorem to find the flux of \( \mathbf{F} = (xy^2, x^2y, y) \) outward through the surface of the region enclosed by the cylinder \( x^2 + y^2 = 1 \) and the planes \( z = -1, \) and \( z = 1. \)

Solution: Recall: \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) \, dv. \) We start with

\[
\nabla \cdot \mathbf{F} = \partial_x(xy^2) + \partial_y(x^2y) + \partial_z(y) \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = y^2 + x^2.
\]

The integration region is \( D = \{ x^2 + y^2 \leq 1, \ z \in [-1, 1] \}. \) So,

\[
I = \iiint_D (\nabla \cdot \mathbf{F}) \, dv = \iiint_D (x^2 + y^2) \, dx \, dy \, dz.
\]

We use cylindrical coordinates,

\[
I = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 \, dz \, r \, dr \, d\theta = 2\pi \left[ \int_0^1 r^3 \, dr \right] (2) = 4\pi \left( \frac{r^4}{4} \right)_0^1.
\]

We conclude that \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \pi. \)

\( \triangle \)

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force \( \mathbf{F} = \langle 2xz, xy, yz \rangle \) along the path \( \mathbf{C} \) given by the intersection of the plane \( x + y + z = 1 \) with the first octant, counterclockwise when viewed from above.

Solution:

Recall: \( \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma. \)

The surface \( S \) is the level surface \( f = 0 \) of \( f = x + y + z - 1 \)

therefore, \( \nabla f = \langle 1, 1, 1 \rangle, \ |\nabla f| = \sqrt{3} \) and \( ||\nabla f \cdot \mathbf{k}|| = 1. \)

\[
\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle, \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy = \sqrt{3} \, dx \, dy.
\]
Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path $C$ given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution: $\mathbf{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$ and $d\sigma = \sqrt{3} dx \, dy$.

We now compute the curl of $\mathbf{F}$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & xy & yz \end{vmatrix} = \langle (z - 0), -(0 - 2x), (y - 0) \rangle$$

so $\nabla \times \mathbf{F} = \langle z, 2x, y \rangle$. Therefore,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \iint_R \left( \langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) \sqrt{3} \, dx \, dy$$

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path $C$ given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:

$$I = \iint_R (z + 2x + y) \, dx \, dy,$$

$z = 1 - x - y$,

$$I = \int_0^1 \int_0^{1-x} (1+x) \, dy \, dx = \int_0^1 (1+x)(1-x) \, dx = \int_0^1 (1-x^2) \, dx.$$

$$I = x \bigg|_0^1 - \frac{x^3}{3} \bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \implies \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3}.$$
Chapter 16, Integration in vector fields.

Example

Find the area of the cone $S$ given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through $S$.

Solution:

Recall: $A(S) = \iint_S d\sigma$. The surface $S$ is the level surface $f = 0$ of the function $f = x^2 + y^2 - z^2$. Also recall that

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy.$$ 

Since $\nabla f = 2\langle x, y, -z \rangle$, we get that

$$|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}, \quad z^2 = x^2 + y^2 \quad \Rightarrow \quad |\nabla f| = 2\sqrt{2}z.$$ 

Also $|\nabla f \cdot \mathbf{k}| = 2z$, therefore, $d\sigma = \sqrt{2} \, dx \, dy$, and then we obtain

$$A(S) = \iint_R \sqrt{2} \, dx \, dy = \int_0^{2\pi} \int_0^1 \sqrt{2}r \, dr \, d\theta = 2\pi \sqrt{2} \frac{r^2}{2} \bigg|_0^1 = \sqrt{2} \pi.$$

Chapter 16, Integration in vector fields.

Example

Find the area of the cone $S$ given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through $S$.

Solution: We now compute the outward flux $I = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$.

Since

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}z} \langle x, y, -z \rangle.$$ 

$$I = \iint_R \frac{1}{\sqrt{2}z} (x^2 + y^2)^{\frac{1}{2}} \, dx \, dy = \iint_R \sqrt{x^2 + y^2} \, dx \, dy.$$ 

Using polar coordinates, we obtain

$$I = \int_0^{2\pi} \int_0^1 r \, r \, dr \, d\theta = 2\pi \frac{r^3}{3} \bigg|_0^1 \quad \Rightarrow \quad I = \frac{2\pi}{3}.$$
Review for Final Exam.

▶ Chapter 16, Sections 16.1-16.8.
▶ **Chapter 15, Sections 15.1-15.5, 15.7.**
▶ Chapter 14, Sections 14.1-14.7.
▶ Chapter 13, Sections 13.1-13.3.
▶ Chapter 12, Sections 12.1-12.6.

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**Chapter 15, Multiple integrals.**

**Example**

Find the volume of the region bounded by the paraboloid \( z = 1 - x^2 - y^2 \) and the plane \( z = 0 \).

**Solution:**

So, \( D = \{ x^2 + y^2 \leq 1, \ 0 \leq z \leq 1 - x^2 - y^2 \} \), and \( R = \{ x^2 + y^2 \leq 1, \ z = 0 \} \). We know that

\[
V(D) = \iiint_D dv = \iint_R \int_0^{1-x^2-y^2} dz \ dx \ dy.
\]

Using cylindrical coordinates \((r, \theta, z)\), we get

\[
V(D) = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} dz \ r \ dr \ d\theta = 2\pi \int_0^1 (1 - r^2) r \ dr.
\]

Substituting \( u = 1 - r^2 \), so \( du = -2r \ dr \), we obtain

\[
V(D) = 2\pi \int_0^0 u \left(\frac{-du}{2}\right) = \pi \left[ u^2 \right]_0^1 \Rightarrow \ V(D) = \frac{\pi}{2}.
\]
Chapter 15, Multiple integrals.

Example
Set up the integrals needed to compute the average of the function $f(x, y, z) = z \sin(x)$ on the bounded region $D$ in the first octant bounded by the plane $z = 4 - 2x - y$. Do not evaluate the integrals.

Solution: Recall: $\bar{f} = \frac{1}{V(D)} \iiint_D f \, dv$.

Since $V(D) = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz \, dy \, dx$,
we conclude that

$$\bar{f} = \frac{\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z \sin(x) \, dz \, dy \, dx}{\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz \, dy \, dx}.$$ 

Chapter 15, Multiple integrals.

Example
Reverse the order of integration and evaluate the double integral $I = \int_0^4 \int_{y/2}^2 e^{x^2} \, dx \, dy$.

Solution: We see that $y \in [0, 4]$ and $x \in [0, y/2]$, that is, $y = 2x$.

Therefore, reversing the integration order means

$$I = \int_0^2 \int_0^{2x} e^{x^2} \, dy \, dx.$$ 

This integral is simple to compute,

$$I = \int_0^2 e^{x^2} \, dx, \quad u = x^2, \quad du = 2x \, dx,$$

$$I = \int_0^4 e^u \, du \quad \Rightarrow \quad I = e^4 - 1.$$