MTH 234 Final Exam Review.

- Monday, December 12, 10:00am - 12:00 noon. (2 hours.)
- Be sure you know where your exam takes place.
- Coverage: Chapters 12-16.
  - Chapter 12, Sections 12.1-12.6.
  - Chapter 13, Sections 13.1-13.3.
  - Chapter 14, Sections 14.1-14.7.
  - Chapter 15, Sections 15.1-15.5, 15.7.
  - Chapter 16, Sections 16.1-16.8.
- From 10 to 20 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

MTH 234 Final Exam Review.

Plan for today:

- Practice final exam April 30, 2001.
- Problems from practice final exam December 11, 2001.
- Extra review problems on Chapters 16, 15.

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- Extra review problems on Chapters 16, 15.

Practice final exam April 30, 2001. Prbl. 1.

Example

Given $A = (1, 2, 3)$, $B = (6, 5, 4)$ and $C = (8, 9, 7)$, find the following:

- $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

  Solution: $\overrightarrow{AB} = \langle (6-1), (5-2), (4-3) \rangle$, hence $\overrightarrow{AB} = \langle 5, 3, 1 \rangle$. In the same way $\overrightarrow{AC} = \langle 7, 7, 4 \rangle$.
- $\overrightarrow{AB} + \overrightarrow{AC} = \langle 12, 10, 5 \rangle$.
- $\overrightarrow{AB} \cdot \overrightarrow{AC} = 35 + 21 + 4$.
- $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 1 \\ 7 & 7 & 4 \end{vmatrix} = \langle (12-7), -(20-7), (35-21) \rangle$,

  hence $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 5, -13, 14 \rangle$.

Practice final exam April 30, 2001. Prbl. 2.

Example

Find the parametric equation of the line through the point $(1, 0, -1)$ and perpendicular to the plane $2x - 3y + 5x = 35$. Then find the intersection of the line and the plane.

Solution: The normal vector to the plane $\langle 2, -3, 5\rangle$ is the tangent vector to the line. Therefore,

$$\mathbf{r}(t) = \langle 1, 0, -1\rangle + t\,\langle 2, -3, 5\rangle,$$

so the parametric equations of the line are

$$x(t) = 1 + 2t, \quad y(t) = -3t, \quad z(t) = -1 + 5t.$$

The intersection point has a t solution of

$$2(1+2t) - 3(-3t) + 5(-1+5t) = 35 \quad \Rightarrow \quad 2 + 4t + 9t - 5 + 25t = 35$$

$$38t = 38 \quad \Rightarrow \quad t = 1 \quad \Rightarrow \quad \mathbf{r}(1) = \langle 3, -3, 4\rangle.$$

Practice final exam April 30, 2001. Prbl. 3.

Example

The velocity of a particle is given by $\mathbf{v}(t) = \langle t^2, (t^3 + 1)\rangle$, and the particle is at $\langle 2, 1\rangle$ for $t = 0$.

- Where is the particle at $t = 2$?

  Solution: $\mathbf{r}(t) = \Big\langle \Big(\dfrac{t^3}{3} + r_x\Big), \Big(\dfrac{t^4}{4} + t + r_y\Big)\Big\rangle$. Since $\mathbf{r}(0) = \langle 2, 1\rangle$, we get that $\mathbf{r}(t) = \Big\langle \Big(\dfrac{t^3}{3} + 2\Big), \Big(\dfrac{t^4}{4} + t + 1\Big)\Big\rangle$. Hence $\mathbf{r}(2) = \langle 8/3 + 2, 7\rangle$.

- Find an expression for the particle arc length for $t \in [0, 2]$.

  Solution: $s(t) = \displaystyle\int_0^t \sqrt{\tau^4 + (\tau^3 + 1)^2}\, d\tau$.

- Find the particle acceleration.

  Solution: $\mathbf{a}(t) = \langle 2t, 3t^2\rangle$.

Practice final exam April 30, 2001. Prbl. 4.

Example

- Draw a rough sketch of the surface $z = 2x^2 + 3y^2 + 5$.

  Solution: This is a paraboloid along the vertical direction, opens up, with vertex at $z = 5$ on the z-axis, and the x-radius is a bit longer than the y-radius.

- Find the equation of the tangent plane to the surface at the point $(1, 1, 10)$.

  Solution: Introduce $f(x, y) = 2x^2 + 3y^2 + 5$, then

  $$L_{(1,1)}(x, y) = \partial_x f(1, 1)\,(x - 1) + \partial_y f(1, 1)\,(y - 1) + f(1, 1).$$

  Since $f(1, 1) = 10$, and $\partial_x f = 4x$, $\partial_y f = 6y$, then

  $$z = L_{(1,1)}(x, y) = 4(x - 1) + 6(y - 1) + 10.$$

Practice final exam April 30, 2001. Prbl. 5.

Example

Let $w = f(x, y)$ and $x = s^2 + t^2$, $y = st^2$. If $\partial_x f = x - y$ and $\partial_y f = y - x$, find $\partial_s w$ and $\partial_t w$ in terms of s and t.

Solution:

$$\partial_s w = \partial_x f\,\partial_s x + \partial_y f\,\partial_s y = (x - y)2s + (y - x)t^2 = (x - y)(2s - t^2).$$

Therefore, $\partial_s w = (s^2 + t^2 - st^2)(2s - t^2)$.

$$\partial_t w = \partial_x f\,\partial_t x + \partial_y f\,\partial_t y = (x - y)2t + (y - x)2st = (x - y)(2t - 2st).$$

Therefore, $\partial_t w = (s^2 + t^2 - st^2)2t(1 - s)$.

Practice final exam April 30, 2001. Prbl. 6.

Example

Find all critical points of the function $f(x,y) = 2x^2 + 8xy + y^4$ and determine whether they re local maximum, minimum of saddle points.

Solution:

$$\nabla f = \langle (4x + 8y), (8x + 4y^3) \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad \begin{cases} x + 2y = 0, \\ 2x + y^3 = 0. \end{cases}$$

$$-4y + y^3 = 0 \Rightarrow \begin{cases} y = 0 \Rightarrow x = 0 & \Rightarrow \quad P_0 = (0,0) \\ y = \pm 2 \Rightarrow x = \mp 4 & \Rightarrow \quad \begin{cases} P_1 = (4, -2) \\ P_2 = (-4, 2) \end{cases} \end{cases}$$

Since $f_{xx} = 4$, $f_{yy} = 12y^2$, and $f_{xy} = 8$, we conclude that $D = 3(16)y^2 - 4(16)$.

Practice final exam April 30, 2001. Prbl. 6.

Example

Find all critical points of the function $f(x,y) = 2x^2 + 8xy + y^4$ and determine whether they re local maximum, minimum of saddle points.

Solution:

$P_0 = (0,0)$, $P_1 = (4,-2)$, $P_2 = (-4,2)$, $D = 3(16)y^2 - 4(16)$.

$$D(0,0) = -4(16) < 0 \quad \Rightarrow \quad P_0 = (0,0) \text{ saddle point.}$$

$$D(4,-2) = 12(16) - 4(16) > 0, \quad f_{xx} = 4 \Rightarrow P_1 = (4,-2) \text{ min.}$$

$$D(-4,2) = 12(16) - 4(16) > 0, \quad f_{xx} = 4 \Rightarrow P_1 = (-4,2) \text{ min.}$$

Practice final exam April 30, 2001. Prbl. 7.

Example

Evaluate the integral $I = \int_0^1 \int_x^{\sqrt{x}} y\, dy\, dx$ by reversing the order of integration.

Solution: The integration region is the set in the square $[0,1] \times [0,1]$ in between the curves $y = x$ and $y = \sqrt{x}$. Therefore,

$$I = \int_0^1 \int_{y^2}^{y} y\, dx\, dy = \int_0^1 y(y - y^2)\, dy = \int_0^1 (y^2 - y^3)\, dy$$

$$I = \Big(\frac{y^3}{3} - \frac{y^4}{4}\Big)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} \quad \Rightarrow \quad I = \frac{1}{12}.$$

Practice final exam April 30, 2001. Prbl. 8.

Example

Find the work done by the force $\mathbf{F} = \langle yz, xz, -xy\rangle$ on a particle moving along the path $\mathbf{r}(t) = \langle t^3, t^2, t\rangle$ for $t \in [0,2]$.

Solution:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{F}(t) = \langle t^3, t^4, -t^5\rangle$ and $\mathbf{r}'(t) = \langle 3t^2, 2t, 1\rangle$. Hence

$$W = \int_0^2 (3t^5 + 2t^5 - t^5)\, dt = \int_0^2 4t^5\, dt = \frac{4}{6}\, t^6\Big|_0^2 = \frac{2}{3}\, 2^6.$$

Therefore, $W = 2^7/3$.

Practice final exam April 30, 2001. Prbl. 9.

Example

Show that the force field
$\mathbf{F} = \langle (y\cos(z) - yze^x), (x\cos(z) - ze^x), (-xy\sin(z) - ye^x)\rangle$ is conservative. Then find its potential function. Then evaluate
$I = \int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{r}(t) = \langle t, t^2, \pi t^3\rangle$ for $t \in [0, 1]$.

Solution: The field $\mathbf{F}$ is conservative, since

$$\partial_x F_y = \cos(z) - ze^x = \partial_y F_x,$$

$$\partial_x F_z = -xy\sin(z) - ye^x = \partial_z F_x,$$

$$\partial_y F_z = -x\sin(z) - e^x = \partial_z F_y.$$

The potential function is a scalar function f solution of

$$\partial_x f = y\cos(z) - yze^x, \ \partial_y f = x\cos(z) - ze^x, \ \partial_z f = -xy\sin(z) - ye^x.$$

Practice final exam April 30, 2001. Prbl. 9.

Example

Show that the force field
$\mathbf{F} = \langle (y\cos(z) - yze^x), (x\cos(z) - ze^x), (-xy\sin(z) - ye^x)\rangle$ is conservative. Then find its potential function. Then evaluate
$I = \int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{r}(t) = \langle t, t^2, \pi t^3\rangle$ for $t \in [0, 1]$.

Solution: Recall:

$$\partial_x f = y\cos(z) - yze^x, \ \partial_y f = x\cos(z) - ze^x, \ \partial_z f = -xy\sin(z) - ye^x.$$

The x-integral of the first equation implies
$f = xy\cos(z) - yze^x + g(y, z)$. Introduce f into the second equation above,

$$x\cos(z) - ze^x + \partial_y g = x\cos(z) - ze^x \quad \Rightarrow \quad \partial_y g(y, z) = 0,$$

so we conclude $g(y, z) = h(z)$, hence $f = xy\cos(z) - yze^x + h(z)$.

Practice final exam April 30, 2001. Prbl. 9.

Example

Show that the force field $\mathbf{F} = \langle (y\cos(z) - yze^x), (x\cos(z) - ze^x), (-xy\sin(z) - ye^x) \rangle$ is conservative. Then find its potential function. Then evaluate $I = \int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{r}(t) = \langle t, t^2, \pi t^3 \rangle$ for $t \in [0, 1]$.

Solution: Recall: $f = xy\cos(z) - yze^x + h(z)$.

Introduce f into the equation $\partial_z f = -xy\sin(z) - ye^x$, that is,

$$-xy\sin(z) - e^x + h'(z) = -xy\sin(z) - ye^x \quad \Rightarrow \quad h'(z) = 0.$$

So, $h(z) = c$, a constant, hence $f = xy\cos(z) - yze^x + c$.

Finally $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,1,\pi)} df = f(1, 1, \pi) - f(0, 0, 0)$.

So we conclude that $\int_C \mathbf{F} \cdot d\mathbf{r} = -(1 + \pi e)$.

Practice final exam April 30, 2001. Prbl. 10.

Example

Use the Green Theorem to evaluate the integral $\int_C F_x\, dx + F_y\, dy$ where $F_x = y + e^x$ and $F_y = 2x^2 + \cos(y)$ and C is the triangle with vertices $(0, 0)$, $(0, 2)$ and $(1, 1)$ traversed counterclockwise.

Solution: Denoting $\mathbf{F} = \langle F_x, F_y \rangle$, Green's Theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_S (\partial_x F_y - \partial_y F_x)\, dA.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (4x - 1)\, dx\, dy = \int_0^1 \int_y^{2-y} (4x - 1)\, dx\, dy.$$

A straightforward calculation gives $\int_C \mathbf{F} \cdot d\mathbf{r} = 3$.

Practice final exam April 30, 2001. Prbl. 11.

Example

Find the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$. Use polar coordinates to evaluate the integral.

Solution:

$$A(S) = \iint_S d\sigma, \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$$

where $f = x^2 + y^2 + z - 4$. Therefore,

$$\nabla f = \langle 2x, 2y, 1\rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{1 + 4x^2 + 4y^2}, \quad \nabla f \cdot \mathbf{k} = 1.$$

$$A(S) = \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta, \quad u = 1 + 4r^2, \quad du = 8r\, dr.$$

The finally obtain $A(S) = (\pi/6)(17^{3/2} - 1)$.

Practice final exam April 30, 2001. Prbl. 12.

Example

Use the Stokes Theorem to evaluate $I = \iint_S \big[\nabla \times (y\mathbf{i})\big] \cdot \mathbf{n}\, d\sigma$ where S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geqslant 0$.

Solution: $\mathbf{F} = \langle y, 0, 0\rangle$. The border of the hemisphere is given by the circle $x^2 + y^2 = 1$, with $z = 0$. This circle can be parametrized for $t \in [0, 2\pi]$ as

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0\rangle,$$

and we also have $\mathbf{F}(t) = \langle \sin(t), 0, 0\rangle$. Therefore,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = -\int_0^{2\pi} \sin^2(t)\, dt$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\frac{1}{2}\int_0^{2\pi} \big[1 - \cos(2t)\big]\, dt.$$

Practice final exam April 30, 2001. Prbl. 12.

Example

Use the Stokes Theorem to evaluate $I = \iint_S \big[\nabla \times (y\,\mathbf{i})\big] \cdot \mathbf{n}\, d\sigma$ where S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geqslant 0$.

Solution: $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\frac{1}{2} \int_0^{2\pi} \big[1 - \cos(2t)\big]\, dt.$

Recall that

$$\int_0^{2\pi} \cos(2t)\, dt = \frac{1}{2} \left(\sin(2t)\Big|_0^{2\pi} \right) = 0.$$

Therefore, we obtain

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\pi.$$

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Practice final exam December 11, 2001. Prbl. 2.

Example

A particle moves along a curve with velocity function $\mathbf{v} = \mathbf{i} + \sqrt{2}\,t\,\mathbf{j} + t^2\,\mathbf{k}$ for $t \in [1, 2]$.

(a) if $\mathbf{r}(1) = \langle 0, \sqrt{2}/2, 2/3 \rangle$, find $\mathbf{r}(t)$.

$\mathbf{r}(t) = (t + c_x)\mathbf{i} + \Big(\frac{\sqrt{2}}{2}\,t^2 + c_y\Big)\mathbf{j} + \Big(\frac{t^3}{3} + c_z\Big)\mathbf{k}.$

$\mathbf{r}(1) = \Big\langle 1 + c_x, \frac{\sqrt{2}}{2} + c_y, \frac{1}{3} + c_z \Big\rangle = \Big\langle 0, \frac{\sqrt{2}}{2}, \frac{2}{3} \Big\rangle$

$c_x = -1,\ \ c_y = 0,\ c_z = 1/3.$ (You finish.)

(b) Distance traveled by the particle from $t = 1$ to $t = 2$.

Arc length: $d = \int_1^2 |\mathbf{v}(t)|\,dt = \int_1^2 \sqrt{1 + 2t^2 + t^4}\,dt$

$d = \int_1^2 \sqrt{(1 + t^2)^2}\,dt = \int_1^2 (1 + t^2)\,dt.$ (You finish.)

(c) Find the acceleration. $\mathbf{a} = \sqrt{2}\,\mathbf{j} + 2t\,\mathbf{k}.$ ◁

Practice final exam December 11, 2001. Prbl. 2.

Example

(a) Sketch the integration region of $I = \int_1^e \int_0^{\ln(x)} y\,dy\,dx.$

The integration region is below $y = \ln(x)$, above $y = 0$, for $x \in [1, e]$.

(b) $I = \int_0^1 \int_{e^y}^e y\,dx\,dy.$

(c) Evaluate I.

$$I = \int_0^1 \int_{e^y}^e y\,dx\,dy = \int_0^1 y(e - e^y)\,dy.$$

$$I = e\int_0^1 y\,dy - \int_0^1 ye^y\,dy.$$

(Integrate by parts.) ◁

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Remark on Chapter 16.

Remark: The normal (flux) form of Green's Theorem is a two-dimensional restriction of the Divergence Theorem.

- The Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F})\, dv.$
- Normal form of Green's Thrm: $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S (\nabla \cdot \mathbf{F})\, dA.$

Remark: The tangential (circulation) form of Green's Theorem is a particular case of the Stokes Theorem when C, S are flat ($z = 0$).

- The Stokes Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$
- Tang. form of Green's Thrm: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$

Chapter 16, Integration in vector fields.

Example

Use the Divergence Theorem to find the flux of $\mathbf{F} = \langle xy^2, x^2y, y\rangle$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$, and $z = 1$.

Solution: Recall: $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D (\nabla\cdot\mathbf{F})\,dv$. We start with

$$\nabla\cdot\mathbf{F} = \partial_x(xy^2) + \partial_y(x^2y) + \partial_z(y) \quad\Rightarrow\quad \nabla\cdot\mathbf{F} = y^2 + x^2.$$

The integration region is $D = \{x^2 + y^2 \leqslant 1,\ z \in [-1, 1]\}$. So,

$$I = \iiint_D (\nabla\cdot\mathbf{F})\,dv = \iiint_D (x^2 + y^2)\,dx\,dy\,dz.$$

We use cylindrical coordinates,

$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\,dz\,r\,dr\,d\theta = 2\pi\Big[\int_0^1 r^3\,dr\Big](2) = 4\pi\Big(\frac{r^4}{4}\Big|_0^1\Big).$$

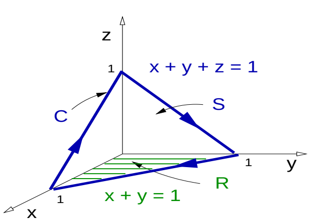
We conclude that $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \pi$. ◁

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz\rangle$ along the path C given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:



Recall: $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$.

The surface S is the level surface $f = 0$ of

$$f = x + y + z - 1$$

therefore, $\nabla f = \langle 1, 1, 1\rangle$, $|\nabla f| = \sqrt{3}$ and $|\nabla f\cdot\mathbf{k}| = 1$.

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle, \qquad d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{k}|}\,dx\,dy = \sqrt{3}\,dx\,dy.$$

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path C given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution: $\mathbf{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$ and $d\sigma = \sqrt{3}\, dx\, dy$.

We now compute the curl of $\mathbf{F}$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & xy & yz \end{vmatrix} = \langle (z - 0), -(0 - 2x), (y - 0) \rangle$$

so $\nabla \times \mathbf{F} = \langle z, 2x, y \rangle$. Therefore,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R \Big(\langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \Big) \sqrt{3}\, dx\, dy$$

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path C given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:

$$I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R \Big(\langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \Big) \sqrt{3}\, dx\, dy.$$

$$I = \iint_R (z + 2x + y)\, dx\, dy, \qquad z = 1 - x - y,$$

$$I = \int_0^1 \int_0^{1-x} (1+x)\, dy\, dx = \int_0^1 (1+x)(1-x)\, dx = \int_0^1 (1-x^2)\, dx.$$

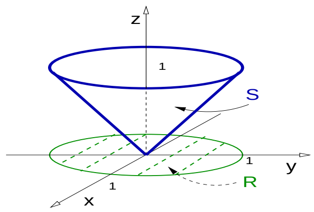
$$I = x\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \quad \Rightarrow \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3}.$$

Chapter 16, Integration in vector fields.

Example

Find the area of the cone S given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through S.

Solution:



Recall: $A(S) = \iint_S d\sigma$. The surface S is the level surface $f = 0$ of the function $f = x^2 + y^2 - z^2$. Also recall that

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy.$$

Since $\nabla f = 2\langle x, y, -z \rangle$, we get that

$$|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}, \quad z^2 = x^2 + y^2 \quad \Rightarrow \quad |\nabla f| = 2\sqrt{2}\, z.$$

Also $|\nabla f \cdot \mathbf{k}| = 2z$, therefore, $d\sigma = \sqrt{2}\, dx\, dy$, and then we obtain

$$A(S) = \iint_R \sqrt{2}\, dx\, dy = \int_0^{2\pi} \int_0^1 \sqrt{2} r\, dr\, d\theta = 2\pi\sqrt{2}\frac{r^2}{2}\Big|_0^1 = \sqrt{2}\,\pi.$$

Chapter 16, Integration in vector fields.

Example

Find the area of the cone S given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through S.

Solution: We now compute the outward flux $I = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$.

Since

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}\, z}\langle x, y, -z \rangle.$$

$$I = \iint_R \frac{1}{\sqrt{2}\, z}(x^2 + y^2)\sqrt{2}\, dx\, dy = \iint_R \sqrt{x^2 + y^2}\, dx\, dy.$$

Using polar coordinates, we obtain

$$I = \int_0^{2\pi} \int_0^1 r\, r\, dr\, d\theta = 2\pi \frac{r^3}{3}\Big|_0^1 \quad \Rightarrow \quad I = \frac{2\pi}{3}.$$

Review for Final Exam.

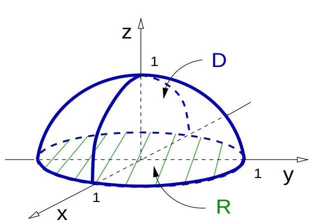
- Chapter 16, Sections 16.1-16.8.
- **Chapter 15, Sections 15.1-15.5, 15.7.**
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1-13.3.
- Chapter 12, Sections 12.1-12.6.

Chapter 15, Multiple integrals.

Example

Find the volume of the region bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:



So, $D = \{x^2 + y^2 \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2 - y^2\}$, and $R = \{x^2 + y^2 \leqslant 1,\ z = 0\}$. We know that

$$V(D) = \iiint_D dv = \iint_R \int_0^{1-x^2-y^2} dz\,dx\,dy.$$

Using cylindrical coordinates (r, θ, z), we get

$$V(D) = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} dz\,r\,dr\,d\theta = 2\pi \int_0^1 (1 - r^2)\,r\,dr.$$

Substituting $u = 1 - r^2$, so $du = -2r\,dr$, we obtain

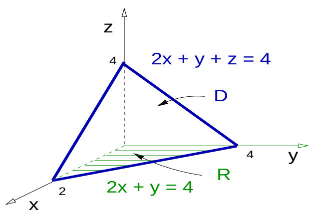
$$V(D) = 2\pi \int_1^0 u\,\frac{(-du)}{2} = \pi \int_0^1 u\,du = \pi \frac{u^2}{2}\Big|_0^1 \quad \Rightarrow \quad V(D) = \frac{\pi}{2}.$$

Chapter 15, Multiple integrals.

Example

Set up the integrals needed to compute the average of the function $f(x, y, z) = z\sin(x)$ on the bounded region D in the first octant bounded by the plane $z = 4 - 2x - y$. Do not evaluate the integrals.

Solution: Recall: $\overline{f} = \dfrac{1}{V(D)} \iiint_D f\, dv$.



Since $V(D) = \displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx$,

we conclude that

$$\overline{f} = \frac{\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z\sin(x)\, dz\, dy\, dx}{\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx}.$$

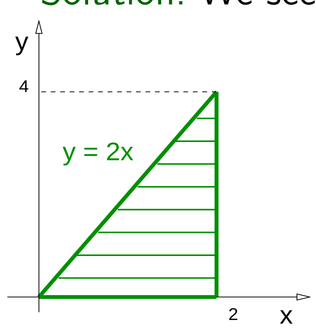
Chapter 15, Multiple integrals.

Example

Reverse the order of integration and evaluate the double integral

$$I = \int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy.$$

Solution: We see that $y \in [0, 4]$ and $x \in [0, y/2]$, that is,



Therefore, reversing the integration order means

$$I = \int_0^2 \int_0^{2x} e^{x^2}\, dy\, dx.$$

This integral is simple to compute,

$$I = \int_0^2 e^{x^2}\, x\, dx, \qquad u = x^2, \quad du = 2x\, dx,$$

$$I = \int_0^4 e^u\, du \quad \Rightarrow \quad I = e^4 - 1.$$