MTH 234 Review for Exam 4

- Sections 16.1-16.8.
- 50 minutes.
- 5 to 10 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4

- **(16.1) Line integrals.**
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area.
- (16.6) Surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Line integrals (16.1)

Example

Integrate the function $f(x,y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0,2]$, from the point $(0,0)$ to $(2,2)$.

Solution: We have to compute $I = \int_C f\, ds$, by that we mean

$$\int_C f\, ds = \int_{t_0}^{t_1} f\big(x(t), y(t)\big)\, |\mathbf{r}'(t)|\, dt,$$

where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \in [t_0, t_1]$ is a parametrization of the path C. In this case the path is given by the parabola $y = x^2/2$, so a simple parametrization is to use $x = t$, that is,

$$\mathbf{r}(t) = \Big\langle t, \frac{t^2}{2}\Big\rangle, \quad t \in [0,2] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, t\rangle.$$

Line integrals (16.1)

Example

Integrate the function $f(x,y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0,2]$, from the point $(0,0)$ to $(2,2)$.

Solution: $\mathbf{r}(t) = \Big\langle t, \frac{t^2}{2}\Big\rangle$ for $t \in [0,2]$, and $\mathbf{r}'(t) = \langle 1, t\rangle$.

$$\int_C f\, ds = \int_{t_0}^{t_1} f\big(x(t), y(t)\big)\, |\mathbf{r}'(t)|\, dt = \int_0^2 \frac{t^3}{t^2/2}\sqrt{1+t^2}\, dt,$$

$$\int_C f\, ds = \int_0^2 2t\sqrt{1+t^2}\, dt, \quad u = 1+t^2, \quad du = 2t\, dt.$$

$$\int_C f\, ds = \int_1^5 u^{1/2}\, du = \frac{2}{3}\, u^{3/2}\Big|_1^5 = \frac{2}{3}\big(5^{3/2} - 1\big).$$

We conclude that $\int_C f\, ds = \frac{2}{3}\big(5\sqrt{5} - 1\big)$. ◁

Review for Exam 4

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- (16.3) Conservative fields, potential functions.
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- (16.6) Surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the work done by the force $\mathbf{F} = \langle yz, zx, -xy \rangle$ in a moving particle along the curve $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0, 2]$.

Solution: The formula for the work done by a force on a particle moving along C given by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt.$$

In this case $\mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle$ for $t \in [0, 2]$. We now need to evaluate $\mathbf{F}$ along the curve, that is,

$$\mathbf{F}(t) = \mathbf{F}\big(x(t), y(t)\big) = \langle t^3, t^4, -t^5 \rangle.$$

Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the work done by the force $\mathbf{F} = \langle yz, zx, -xy \rangle$ in a moving particle along the curve $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0, 2]$.

Solution: $\mathbf{F}(t) = \langle t^3, t^4, -t^5 \rangle$ and. $\mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle$ for $t \in [0, 2]$. The Work done by the force on the particle is

$$W = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = \int_0^2 \langle t^3, t^4, -t^5 \rangle \cdot \langle 3t^2, 2t, 1 \rangle\, dt$$

$$W = \int_0^2 \left(3t^5 + 2t^5 - t^5\right) dt = \int_0^2 4t^5\, dt = \frac{4}{6} t^6 \Big|_0^2 = \frac{2}{3}\, 2^6.$$

We conclude that $W = 2^7/3$.

Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz \rangle$ from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: The flow (also called circulation) of the field $\mathbf{F}$ along a curve C parametrized by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt.$$

We use $t = x$ as the parameter of the curve $\mathbf{r}$, so we obtain

$$\mathbf{r}(t) = \langle t, t^2, t \rangle, \quad t \in [0, 1] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, 2t, 1 \rangle.$$

$$\mathbf{F}(t) = \langle t(t^2), (t^2)^2, -t^2(t) \rangle \quad \Rightarrow \quad \mathbf{F}(t) = \langle t^3, t^4, -t^3 \rangle.$$

Example

Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz\rangle$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: $\mathbf{r}'(t) = \langle 1, 2t, 1\rangle$ for $t \in [0,1]$ and $\mathbf{F}(t) = \langle t^3, t^4, -t^3\rangle$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \langle t^3, t^4, -t^3\rangle \cdot \langle 1, 2t, 1\rangle\, dt,$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^3 + 2t^5 - t^3\right) dt = \int_0^1 2t^5\, dt = \frac{2}{6}\, t^6\Big|_0^1.$$

We conclude that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}$. ◁

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x - y)\rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $t \in [0, 2\pi]$.

Solution: The flux (also normal flow) of the field $\mathbf{F} = \langle F_x, F_y\rangle$ across a loop C parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \in [t_0, t_1]$ is given by

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \left[\mathbf{F}_x y'(t) - F_y x'(t)\right] dt.$$

Recall that $\mathbf{n} = \dfrac{1}{|\mathbf{r}'(t)|}\, \langle y'(t), -x'(t)\rangle$ and $ds = |\mathbf{r}'(t)|\, dt$, therefore

$$\mathbf{F} \cdot \mathbf{n}\, ds = \Big(\langle F_x, F_y\rangle \cdot \frac{1}{|\mathbf{r}'(t)|}\, \langle y'(t), -x'(t)\rangle\Big)\, |\mathbf{r}'(t)|\, dt,$$

so we obtain $\mathbf{F} \cdot \mathbf{n}\, ds = \left[\mathbf{F}_x y'(t) - F_y x'(t)\right] dt$.

Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \big[\mathbf{F}_x y'(t) - F_y x'(t)\big]\,dt.$

We evaluate $\mathbf{F}$ along the loop,

$$\mathbf{F}(t) = \langle -a\cos(t), a\big[\cos(t) - \sin(t)\big] \rangle,$$

and compute $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$. Therefore,

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a\cos(t)a\cos(t) - a\big(\cos(t) - \sin(t)\big)(-a)\sin(t)\big]\,dt$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t)\big]\,dt$$

Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t)\big]\,dt.$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = a^2\int_0^{2\pi} \big[-1 + \sin(t)\cos(t)\big]\,dt,$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = a^2\int_0^{2\pi} \Big[-1 + \frac{1}{2}\sin(2t)\Big]\,dt.$$

Since $\displaystyle\int_0^{2\pi}\sin(2t)\,dt = 0$, we obtain $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = -2\pi a^2$. $\triangleleft$

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Conservative fields, potential functions (16.3)

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x \cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y \cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field, that means there exists a scalar field f such that $\mathbf{F} = \nabla f$. The equations for f are

$$\partial_x f = y \sin(z), \quad \partial_y f = x \sin(z), \quad \partial_z f = xy \cos(z).$$

Conservative fields, potential functions (16.3)

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = c.$$

We conclude that $f(x,y,z) = xy\sin(z) + c$. $\triangleleft$

Conservative fields, potential functions (16.3)

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle\cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0,1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz\big]$$

$$I = f(1,2,\pi/2) - f(1,1,\pi/2) = 2\sin(\pi/2) - \sin(\pi/2) \;\Rightarrow\; I = 1.$$

Conservative fields, potential functions (16.3)

Example

Show that the differential form in the integral below is exact,

$$\int_C \Big[3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz\Big], \qquad y > 0.$$

Solution: We need to show that the field $\mathbf{F} = \Big\langle 3x^2, \frac{z^2}{y}, 2z\ln(y)\Big\rangle$ is conservative. It is, since,

$$\partial_y F_z = \frac{2z}{y} = \partial_z F_y, \quad \partial_x F_z = 0 = \partial_z F_x, \quad \partial_x F_y = 0 = \partial_y F_x.$$

Therefore, exists a scalar field f such that $\mathbf{F} = \nabla f$, or equivalently,

$$df = 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz.$$

Conservative fields, potential functions (16.3)

Example

Compute $I = \int_{(0,0,0)}^{(1,-1,0)} 2x\cos(z)\,dx + z\,dy + (y - x^2\sin(z))\,dz.$

Solution: The integral is specified by the path end points. That suggests that the vector field is a gradient field.

$$\mathbf{F} = \langle 2x\cos(z), z, [y - x^2\sin(z)]\rangle = \nabla f = \langle \partial_x f, \partial_y f, \partial_z f\rangle.$$

$$\partial_x f = 2x\cos(z) \quad \Rightarrow \quad f = x^2\cos(z) + g(y,z).$$

$$\partial_y f = z = \partial_y g \;\Rightarrow\; g = yz + h(z) \;\Rightarrow\; f = x^2\cos(z) + yz + h(z).$$

$$\partial_z f = y - x^2\sin(z) = -x^2\sin(z) + y + h' \quad \Rightarrow \quad h' = 0$$

Since $f = x^2\cos(z) + yz + c$, we obtain

$$I = \int_{(0,0,0)}^{(1,-1,0)} \nabla f \cdot d\mathbf{r} = f(1,-1,0) - f(0,0,0) \quad \Rightarrow \quad I = 1. \triangleleft$$

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The Green Theorem in a plane (16.4)

Example

Use the Green Theorem in the plane to evaluate the line integral given by $\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big]$ on the circle C defined by $(x-1)^2 + (y-3)^2 = 4$.

Solution: $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\,dx\,dy$; and $\operatorname{curl}\mathbf{F} = \big(\partial_x F_y - \partial_y F_x\big)$.

Here $\mathbf{F} = \langle (6y + x), (y + 2x)\rangle$. Since $\partial_x F_y = 2$ and $\partial_y F_x = 6$, hence $\operatorname{curl}\mathbf{F} = 2 - 6 = -4$. Green's Theorem implies

$$\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big] = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (-4)\,dx\,dy.$$

Since the area of the disk $S = \{(x-1)^2 + (y-3)^2 \leqslant 4\}$ is $\pi(2^2)$,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -4\iint_S dx\,dy = -4(4\pi) \quad \Rightarrow \quad \oint_C \mathbf{F}\cdot d\mathbf{r} = -16\pi.$$

The Green Theorem in a plane (16.4)

Example

Use the Green Theorem in the plane to find the flux of
$\mathbf{F} = (x - y^2)\,\mathbf{i} + (x^2 + y)\,\mathbf{j}$ through the ellipse $9x^2 + 4y^2 = 36$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S \operatorname{div} \mathbf{F}\, dx\, dy.$

Recall: $\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y$. Here is simpler to compute the right-hand side than the left-hand side. $\operatorname{div} \mathbf{F} = 1 + 1 = 2$.
Green's Theorem implies

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (2)\, dx\, dy. = 2\, A(R).$$

Since R is the ellipse $x^2/4 + y^2/9 = 1$, its area is $A(R) = (2)(3)\pi$.
We conclude

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 12\pi. \qquad \triangleleft$$

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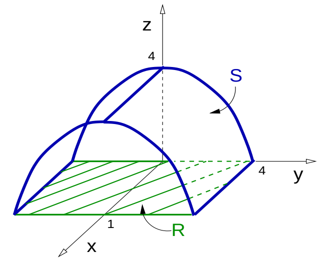
Surface area (16.5)

Example

Set up the integral for the area of the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$, $z = 0$.

Solution:



We must compute: $A(S) = \iint_S d\sigma.$

Recall $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$, with $\mathbf{k} \perp R$.

Recall: $f(x, y, z) = y^2 + 4z - 16.$

$$\nabla f = \langle 0, 2y, 4\rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}.$$

Since $R = [0, 1] \times [-4, 4]$, its normal vector is $\mathbf{k}$ and $|\nabla f \cdot \mathbf{k}| = 4$. Then,

$$A(S) = \iint_R \frac{2\sqrt{4 + y^2}}{4}\, dx\, dy \Rightarrow A(S) = \int_0^1 \int_{-4}^4 \frac{2\sqrt{4 + y^2}}{4}\, dy\, dx.$$

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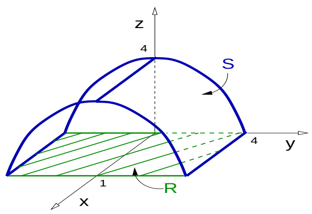
Surface integrals (16.6)

Example

Integrate the function $g(x, y, z) = x\sqrt{4 + y^2}$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution:



We must compute: $I = \iint_S g \, d\sigma$.

Recall $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dx\, dy$, with $\mathbf{k} \perp R$

and in this case $f(x, y, z) = y^2 + 4z - 16$.

$$\nabla f = \langle 0, 2y, 4\rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}.$$

Since $R = [0, 1] \times [-4, 4]$, its normal vector is $\mathbf{k}$ and $|\nabla f \cdot \mathbf{k}| = 4$. Then,

$$\iint_S g \, d\sigma = \iint_R \left(x\sqrt{4 + y^2}\right) \frac{2\sqrt{4 + y^2}}{4} \, dx\, dy.$$

Surface integrals (16.6)

Example

Integrate the function $g(x, y, z) = x\sqrt{4 + y^2}$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution: $\iint_S g \, d\sigma = \iint_R \left(x\sqrt{4 + y^2}\right) \dfrac{2\sqrt{4 + y^2}}{4} \, dx\, dy.$

$$\iint_S g \, d\sigma = \frac{1}{2} \iint_R x(4 + y^2)\, dx\, dy = \frac{1}{2} \int_{-4}^{4} \int_0^1 x(4 + y^2)\, dx\, dy$$

$$\iint_S g \, d\sigma = \frac{1}{2} \Big[\int_{-4}^{4} (4 + y^2)\, dy\Big] \Big[\int_0^1 x\, dx\Big] = \frac{1}{2} \Big(4y + \frac{y^3}{3}\Big)\Big|_{-4}^{4} \Big(\frac{x^2}{2}\Big)\Big|_0^1$$

$$\iint_S g \, d\sigma = \frac{1}{2} 2\Big(4^2 + \frac{4^3}{3}\Big)\frac{1}{2} = 8\Big(1 + \frac{4}{3}\Big) \quad \Rightarrow \quad \iint_S g \, d\sigma = \frac{56}{3}.$$

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The Stokes Theorem (16.7)

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and $S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

Solution: Recall: $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

The surface S is the cylinder walls and its cover at $z = h$. Therefore, the curve C is the circle $x^2 + y^2 = a^2$ at $z = 0$. That circle can be parametrized (counterclockwise) as $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$.

The Stokes Theorem (16.7)

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
$S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

Solution: $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$. Hence

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \big(a^2 \sin^2(t) + a^2\cos^2(t)\big)\, dt = \int_0^{2\pi} a^2\, dt.$$

We conclude that $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 2\pi a^2$.

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- **(16.8) The Divergence Theorem.**

The Divergence Theorem (16.8)

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution: Recall: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv.$

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 3x.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv = \iint_D 3x\, dx\, dy\, dz.$$

It is convenient to use spherical coordinates:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \left[3\rho \sin(\phi)\cos(\phi)\right] \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

The Divergence Theorem (16.8)

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

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$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \left[\int_0^{\pi/2} \cos(\theta)\, d\theta\right] \left[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\right] \left[\int_0^2 3\rho^3\, d\rho\right]$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \left[\sin(\theta)\Big|_0^{\pi/2}\right] \left[\frac{1}{2}\int_0^{\pi/2} \big(1 - \cos(2\phi)\big)\, d\phi\right] \left[\frac{3}{4}\rho^4\Big|_0^2\right]$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = (1)\frac{1}{2}\left(\frac{\pi}{2}\right)(12) \quad \Rightarrow \quad \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 3\pi.$$