# Review for Exam 3

- Tuesday Recitations: 14.7, 15.1-15.5, half 15.7.
- Thursday Recitations: 15.1-15.5, 15.7.
- 50 minutes.
- From five 10-minute problems to ten 5-minutes problems.
- Problems similar to homework problems.
- No calculators, no notes, no books, no phones.

# Double integrals in Cartesian coordinates (Section 15.2)

Example

Switch the integration order in $I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx$.

Solution:

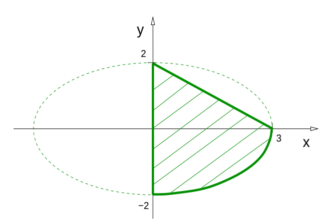
We first draw the integration region. Start with the outer limits.

$x \in [0,3]$.

$y \leqslant 2 - 2x/3$ and $y \geqslant 2\sqrt{1 - \dfrac{x^2}{3^2}}$.

The lower limit is part of the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$$

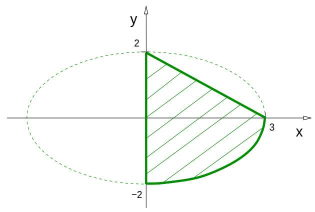


## Double integrals in Cartesian coordinates (Section 15.2)

**Example**

Switch the integration order in $I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx$.

**Solution:**




Split the integral at $y = 0$.
In $y \in [-2, 0]$, holds $0 \leqslant x$.
The upper limit comes from
$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$,
so, $x = +3\sqrt{1 - \frac{y^2}{2^2}}$.

In $y \in [0, 2]$, holds $0 \leqslant x$. The upper limit comes from $y = 2\Big(1 - \frac{x}{3}\Big)$, that is, $x = 3\Big(1 - \frac{y}{2}\Big)$. We then conclude:

$$I = \int_{-2}^{0} \int_0^{3\sqrt{1-\frac{y^2}{2^2}}} f(x,y)\,dx\,dy + \int_0^2 \int_0^{3(1-\frac{y}{2})} f(x,y)\,dx\,dy.$$

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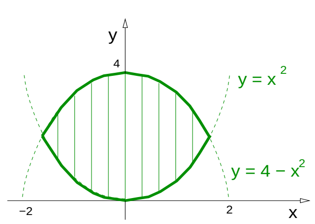
## Areas as double integrals (Section 15.3)

**Example**

Compute the are of the region on the $xy$-plane below the curve $y = 4 - x^2$ and above $y = x^2$. Also switch the integration order.

**Solution:** First, sketch the integration region.




It is simpler integrating $dy\,dx$.

$$A = \int_{-2}^{2} \int_{x^2}^{4-x^2} dy\,dx.$$

$$A = \int_{-2}^{2} \big[(4 - x^2) - x^2\big]\,dx$$

$$A = \int_{-2}^{2} (4 - 2x^2)\,dx = 4x\Big|_{-2}^{2} - \frac{2}{3}x^3\Big|_{-2}^{2} = (8+8) - \frac{2}{3}(8+8)$$

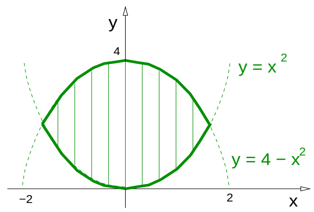
$$A = \frac{16}{3}.$$

# Areas as double integrals (Section 15.3)

## Example

Compute the are of the region on the $xy$-plane below the curve $y = 4 - x^2$ and above $y = x^2$. Also switch the integration order.

Solution: We now interchage the integration region to $dx\,dy$.




We need to divide the $y$-interval at $y$ such that

$$4 - x^2 = x^2 \;\Rightarrow\; x = \pm\sqrt{2}.$$

That is, $y = 2$. Then,

$$A = \int_0^2 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_2^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy.$$

$\triangleleft$

# Double integrals in polar coordinates. (Sect. 15.4)

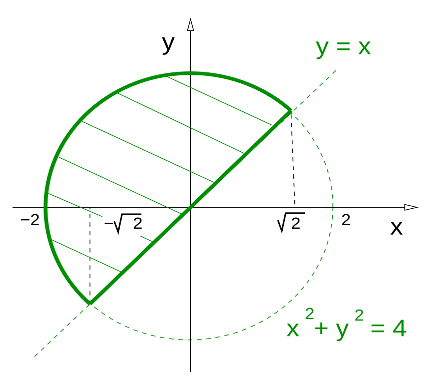
## Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{-\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \big(x^2+y^2\big)\,dy\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} \big(x^2+y^2\big)\,dy\,dx.$$

Solution: First sketch the integration region.

- $x \in [-2, \sqrt{2}]$.
- For $x \in [-2, -\sqrt{2}]$, we have $|y| \leqslant \sqrt{4 - x^2}$, so the curve is part of the circle $x^2 + y^2 = 4$.
- For $x \in [-\sqrt{2}, \sqrt{2}]$, we have that $y$ is between the line $y = x$ and the upper side of the circle $x^2 + y^2 = 4$.



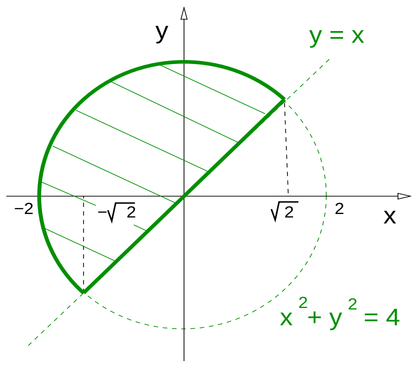
## Double integrals in polar coordinates. (Sect. 15.4)

Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{-\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx.$$

Solution:




$$I = \int_{\pi/4}^{5\pi/4} \int_0^2 r^2\, r dr\, d\theta$$

$$I = \Big(\frac{5\pi}{4} - \frac{\pi}{4}\Big) \int_0^2 r^3\, dr$$

$$I = \pi\Big(\frac{r^4}{4}\Big|_0^2\Big)$$

$$I = 4\pi.$$ ◁

## Double integrals in polar coordinates. (Sect. 15.4)

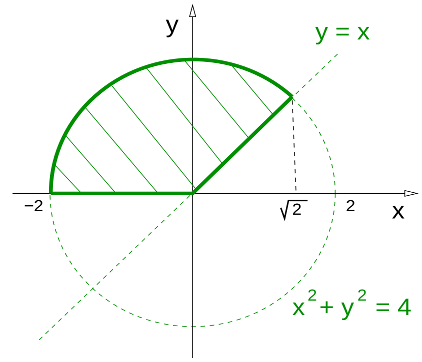
Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx + \int_{0}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$

Solution: First sketch the integration region.

- $x \in [-2, \sqrt{2}]$.
- For $x \in [-2, 0]$, we have $y \geqslant 0$ and $y \leqslant \sqrt{4-x^2}$. The latter curve is part of the circle $x^2 + y^2 = 4$.
- For $x \in [0, \sqrt{2}]$, we have $y \geqslant x$ and $y \leqslant \sqrt{4-x^2}$.



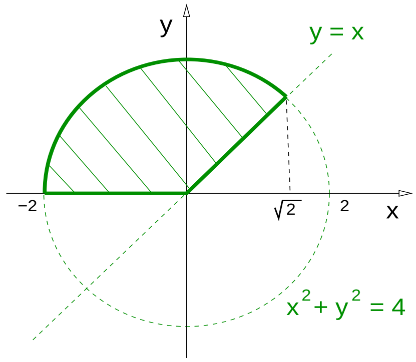
## Double integrals in polar coordinates. (Sect. 15.4)

Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx + \int_{0}^{\sqrt{2}}\int_{x}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$

Solution:




$$I = \int_{\pi/4}^{\pi}\int_{0}^{2} r^2\, r dr\, d\theta$$

$$I = \frac{3\pi}{4}\Big(\frac{r^4}{4}\Big|_0^2\Big)$$

We conclude: $I = 3\pi$. ◁

## Triple integral in Cartesian coordinates (Sect. 15.5)

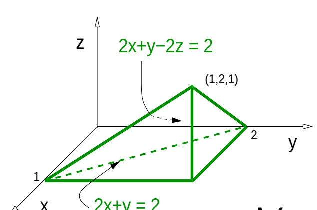
Example

Find the volume of the region in the first octant below the plane $2x + y - 2z = 2$ and $x \leqslant 1$, $y \leqslant 2$.

Solution: First sketch the integration region.

The plane contains the points $(1,0,0)$, $(0,2,0)$, $(1,2,1)$.

We choose the order $dz\,dy\,dx$. The integral is




$$V = \int_0^1\int_{2-2x}^{2}\int_0^{-1+x+y/2} dz\,dy\,dx.$$

$$V = \int_0^1\int_{2-2x}^{2}\Big[(-1+x) + \frac{y}{2}\Big]dy\,dx,$$

$$V = \int_0^1\Big[-(1-x)[2-2(1-x)] + \frac{1}{4}[4-4(1-x)^2]\Big]\,dx.$$

## Triple integral in Cartesian coordinates (Sect. 15.5)

### Example

Find the volume of the region in the first octant below the plane $2x + y - 2z = 2$ and $x \leqslant 1$, $y \leqslant 2$.

Solution: $V = \displaystyle\int_0^1 \Big[ -(1-x)[2 - 2(1-x)] + \frac{1}{4}[4 - 4(1-x)^2] \Big]\, dx.$

$$V = \int_0^1 \big[ -2(1-x) + 2(1-x)^2 + 1 - (1-x)^2 \big]\, dx,$$

$$V = \int_0^1 \big[ -1 + 2x + (1-x)^2 \big]\, dx = \int_0^1 \big[ -1 + 2x + 1 + x^2 - 2x \big]\, dx$$

$$V = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 \quad \Rightarrow \quad V = \frac{1}{3}. \qquad \triangleleft$$

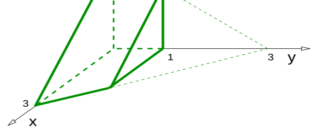
## Triple integral in Cartesian coordinates (Sect. 15.5)

### Example

Find the volume of the region in the first octant below the plane $x + y + z = 3$ and $y \leqslant 1$.

Solution: First sketch the integration region.

The plane contains the points $(1, 0, 0)$, $(0, 2, 0)$, $(1, 2, 1)$.




We choose the order $dz\, dy\, dx$.
We need $x + y = 3$ at $z = 0$.

$$V = \int_0^2 \int_0^1 \int_0^{3-x-y} dz\, dy\, dx$$

$$+ \int_2^3 \int_0^{3-x} \int_0^{3-x-y} dz\, dy\, dx.$$

$$V = \int_0^2 \int_0^1 (3 - x - y)\, dy\, dx + \int_2^3 \int_0^{3-x} (3 - x - y)\, dy\, dx.$$

## Triple integral in Cartesian coordinates (Sect. 15.5)

### Example

Find the volume of the region in the first octant below the plane $x + y + z = 3$ and $y \leqslant 1$.

Solution:

$$V = \int_0^2 \int_0^1 (3 - x - y)\,dy\,dx + \int_2^3 \int_0^{3-x} (3 - x - y)\,dy\,dx.$$

$$V = \int_0^2 \Big[(3-x)\Big(y\Big|_0^1\Big) - \Big(\frac{y^2}{2}\Big|_0^1\Big) + (3-x)\Big(y\Big|_0^{(3-x)}\Big) - \Big(\frac{y^2}{2}\Big|_0^{(3-x)}\Big)\Big]\,dx$$

$$V = \int_0^2 \Big[(3 - x) - \frac{1}{2} + (3 - x)^2 - \frac{1}{2}(3 - x)^2\Big]\,dx$$

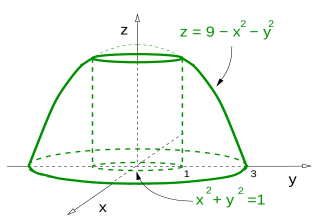
$$V = \int_0^2 \Big[\frac{5}{2} - x + \frac{1}{2}(3 - x)^2\Big]\,dx \quad \Rightarrow \quad V = \frac{22}{3}. \qquad \triangleleft$$

## Triple integral in spherical coordinates (Sect. 15.7)

### Example

Use spherical coordinates to find the volume of the region below the paraboloid $z = 9 - x^2 - y^2$ below the $xy$-plane and outside the cylinder $x^2 + y^2 = 1$.

Solution: First sketch the integration region.




In cylindrical coordinates,

$$z = 9 - x^2 - y^2 \Leftrightarrow z = 9 - r^2.$$

$$x^2 + y^2 = 1 \Leftrightarrow r = 1.$$

$$V = \int_0^{2\pi} \int_1^3 \int_0^{9-r^2} r\,dz\,dr\,d\theta = 2\pi \int_1^3 (9 - r^2)\,r\,dr$$

$$V = 2\pi \Big(\frac{9r^2}{2} - \frac{r^4}{4}\Big)\Big|_1^3 \quad \Rightarrow \quad V = 32\pi. \qquad \triangleleft$$

## Triple integral in spherical coordinates (Sect. 15.7)

### Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the half sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

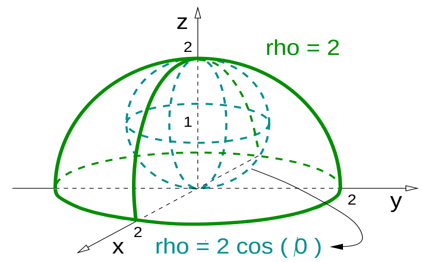
Solution: First sketch the integration region.

- $\rho = 2\cos(\phi)$ is a sphere, since

$$\rho^2 = 2\rho\cos(\phi) \Leftrightarrow x^2 + y^2 + z^2 = 2z$$

$$x^2 + y^2 + (z-1)^2 = 1.$$

- $\rho = 2$ is a sphere radius 2 and $\phi \in [0, \pi/2]$ says we only consider the upper half of the sphere.



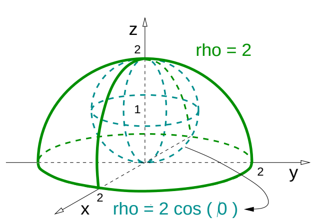

## Triple integral in spherical coordinates (Sect. 15.7)

### Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution:

$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_{2\cos(\phi)}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$




$$V = 2\pi \int_0^{\pi/2} \Big(\frac{\rho^3}{3}\Big|_{2\cos(\phi)}^{2}\Big) \sin(\phi)\, d\phi$$

$$V = \frac{2\pi}{3}\int_0^{\pi/2}\Big[8\sin(\phi) - 8\cos^3(\phi)\,\sin(\phi)\Big]\, d\phi.$$

$$V = \frac{16\pi}{3}\Big[\Big(-\cos(\phi)\Big|_0^{\pi/2}\Big) - \int_0^{\pi/2}\cos^3(\phi)\sin(\phi)\, d\phi\Big].$$

## Triple integral in spherical coordinates (Sect. 15.7)

### Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution: $V = \dfrac{16\pi}{3}\Big[\Big(-\cos(\phi)\Big|_0^{\pi/2}\Big) - \displaystyle\int_0^{\pi/2} \cos^3(\phi)\sin(\phi)\,d\phi\Big].$

Introduce the substitution: $u = \cos(\phi)$, $du = -\sin(\phi)\,d\phi$.

$$V = \frac{16\pi}{3}\Big[1 + \int_1^0 u^3\,du\Big] = \frac{16\pi}{3}\Big[1 + \Big(\frac{u^4}{4}\Big|_1^0\Big)\Big] = \frac{16\pi}{3}\Big(1 - \frac{1}{4}\Big).$$

$$V = \frac{16\pi}{3}\,\frac{3}{4} \quad\Rightarrow\quad V = 4\pi.$$

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## Triple integral in cylindrical coordinates (Sect. 15.7)

### Example

Use cylindrical coordinates to find the volume in the $z \geqslant 0$ region of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

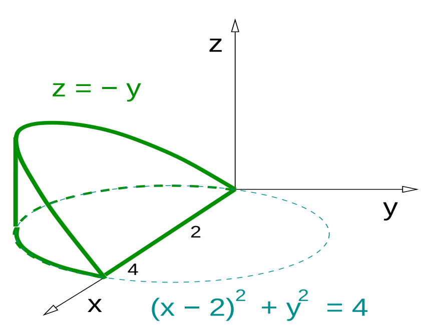
Solution: First sketch the integration region.

- $(x-2)^2 + y^2 = 4$ is a circle in the $xy$-plane, since

$$x^2 + y^2 = 4x \Leftrightarrow r^2 = 4r\cos(\theta)$$

$$r = 4\cos(\theta).$$

- Since $0 \leqslant z \leqslant -y$, the integration region is on the $y \leqslant 0$ part of the $z = 0$ plane.

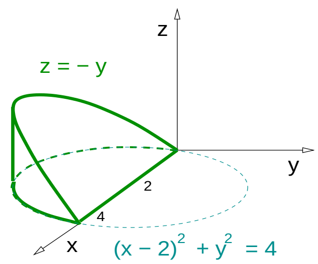


## Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume in the $z \geqslant 0$ region of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution:




$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \int_0^{-r\sin(\theta)} r\,dz\,dr\,d\theta.$$

$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \big[-r\sin(\theta) - 0\big]\,r\,dr\,d\theta$$

$$V = -\int_{3\pi/2}^{2\pi} \Big(\frac{r^3}{3}\Big|_0^{4\cos(\theta)}\Big) \sin(\theta)\,d\theta.$$

$$V = -\int_{3\pi/2}^{2\pi} \frac{4^3}{3} \cos^3(\theta)\sin(\theta)\,d\theta.$$

## Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution: $V = -\displaystyle\int_{3\pi/2}^{2\pi} \frac{4^3}{3} \cos^3(\theta)\sin(\theta)\,d\theta.$

Introduce the substitution: $u = \cos(\theta)$, $du = -\sin(\theta)\,d\theta$;

$$V = \frac{4^3}{3}\int_0^1 u^3\,du = \frac{4^3}{3}\Big(\frac{u^4}{4}\Big|_0^1\Big) = \frac{4^3}{3}\,\frac{1}{4}.$$

We conclude: $V = \dfrac{16}{3}$. $\triangleleft$