Review for Exam 2

- Tuesday Recitations: Sections 13.1-13.3. 14.1-14.6.
- Thursday Recitations: Sections 13.1-13.3. 14.1-14.7.
- 50 minutes.
- From five 10-minute problems to ten 5-minutes problems.
- Problems similar to homework problems.
- No calculators, no notes, no books, no phones.

Section 13.2: Projectile motion

Example

Find the position $\mathbf{r}$ and velocity functions $\mathbf{v}$ of a particle moving with an acceleration $\mathbf{a}(t) = \langle 0, 0, -10\rangle\ m/sec^2$. The initial velocity and position are, $\mathbf{v}(0) = \langle 0, 2, 1\rangle\ m/sec$ and $\mathbf{r}(0) = \langle 0, 0, 2\rangle\ m$.

Solution: Since $\mathbf{a}(t) = \langle 0, 0, -10\rangle$,

$$\left.\begin{array}{r} \mathbf{v}(t) = \langle v_{0x}, v_{0y}, -10\,t + v_{0z}\rangle, \\ \mathbf{v}(0) = \langle 0, 2, 1\rangle, \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} v_{0x} = 0, \\ v_{0y} = 2, \\ v_{0z} = 1. \end{array}\right.$$

The velocity is $\mathbf{v}(t) = \langle 0, 2, -10\,t + 1\rangle$. the position is

$$\left.\begin{array}{r} \mathbf{r}(t) = \langle r_{0x}, 2t + r_{0y}, -5t^2 + t + r_{0z}\rangle, \\ \mathbf{r}(0) = \langle 0, 0, 2\rangle, \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} r_{0x} = 0, \\ r_{0y} = 0, \\ r_{0z} = 2. \end{array}\right.$$

We conclude that $\mathbf{r}(t) = \langle 0, 2t, -5t^2 + t + 2\rangle$. $\triangleleft$

Section 13.2: Projectile motion

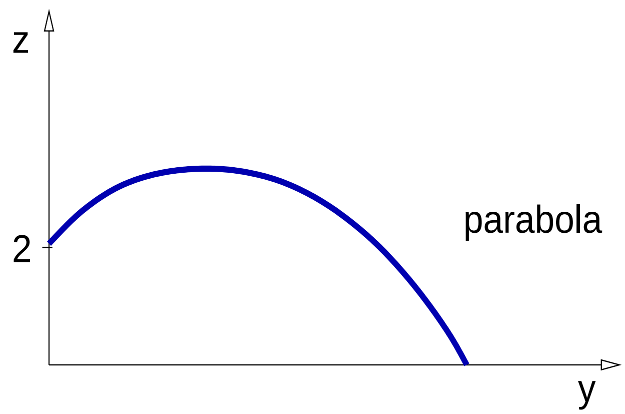
Example

Find the trajectory of the particle in the previous example.

Solution: Recall $\mathbf{r}(t) = \langle 0, 2t, -5t^2 + t + 2 \rangle$.

$$x(t) = 0, \quad y(t) = 2t, \quad z(t) = -5t^2 + t + 2.$$

The trajectory is $z(y) = -\frac{5}{4} y^2 + \frac{y}{2} + 2$.



Section 13.3

Example

Reparametrize with respect to its arclength, starting at $t = 0$, the position function $\mathbf{r}$ corresponding to the acceleration $\mathbf{a}(t) = \langle\, [-\sin(t) - \cos(t)], [\cos(t) - \sin(t)], 0 \rangle$ and initial velocity $\mathbf{v}_0 = \langle 1, 1, 1 \rangle$ and initial position $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$.

Solution: First, we need to find $\mathbf{r}(t)$. Then, we reparametrize.

$$\mathbf{v}(t) = \langle\, [\cos(t) - \sin(t) + v_{0x}], [\sin(t) + \cos(t) + v_{0y}], v_{0z} \rangle.$$

The initial condition implies

$$\langle 1 + v_{0x}, 1 + v_{0y}, v_{0z} \rangle = \langle 1, 1, 1 \rangle \quad \Rightarrow \quad v_{0x} = 0 = v_{0y}, \quad v_{0z} = 1.$$

$$\mathbf{v}(t) = \langle\, [\cos(t) - \sin(t)], [\sin(t) + \cos(t)], 1 \rangle.$$

Section 13.3

Example

Reparametrize with respect to its arclength, starting at $t = 0$, the position function $\mathbf{r}$ corresponding to the acceleration $\mathbf{a}(t) = \langle\, [-\sin(t) - \cos(t)], [\cos(t) - \sin(t)], 0\rangle$ and initial velocity $\mathbf{v}_0 = \langle 1, 1, 1\rangle$ and initial position $\mathbf{r}_0 = \langle 0, 0, 0\rangle$.

Solution: Recall: $\mathbf{v}(t) = \langle\, [\cos(t) - \sin(t)], [\sin(t) + \cos(t)], 1\rangle$.

The position function is

$$\mathbf{r}(t) = \langle\, [\sin(t) + \cos(t) + r_{0x}], [-\cos(t) + \sin(t) + r_{0y}], t + r_{0z}\rangle.$$

The initial condition implies

$$\langle 1 + r_{0x}, -1 + r_{0y}, r_{0z}\rangle = \langle 0, 0, 0\rangle \;\Rightarrow\; r_{0x} = -1,\; r_{0y} = 1,\; r_{0z} = 0.$$

$$\mathbf{r}(t) = \langle\, [\sin(t) + \cos(t) - 1], [-\cos(t) + \sin(t) + 1], t\rangle.$$

Section 13.3

Example

Reparametrize with respect to its arclength, starting at $t = 0$, the position function $\mathbf{r}$.

Solution: Recall: $\mathbf{v}(t) = \langle\, [\cos(t) - \sin(t)], [\sin(t) + \cos(t)], 1\rangle$, and recall $\mathbf{r}(t) = \langle\, [\sin(t) + \cos(t) - 1], [-\cos(t) + \sin(t) + 1], t\rangle$.

Then, $|\mathbf{v}(t)|^2 = [\cos(t) - \sin(t)]^2 + [\cos(t) + \sin(t)]^2 + 1$

$$|\mathbf{v}(t)|^2 = \cos^2(t) + \sin^2(t) - 2\sin(t)\cos(t)$$

$$+\cos^2(t) + \sin^2(t) + 2\sin(t)\cos(t) + 1$$

$|\mathbf{v}(t)| = \sqrt{3}$, therefore $\ell(t) = \displaystyle\int_0^t \sqrt{3}\, d\tau$ implies $\ell = \sqrt{3}\, t$.

$$\hat{\mathbf{r}}(\ell) = \Big\langle \Big[\sin\Big(\frac{\ell}{\sqrt{3}}\Big) + \cos\Big(\frac{\ell}{\sqrt{3}}\Big) - 1\Big], \Big[-\cos\Big(\frac{\ell}{\sqrt{3}}\Big) + \sin\Big(\frac{\ell}{\sqrt{3}}\Big) + 1\Big], \frac{\ell}{\sqrt{3}}\Big\rangle.$$

Section 13.3

Example

Reparametrize the curve $\mathbf{r}(t) = \left\langle \frac{3}{2}\sin(t^2), 2t^2, \frac{3}{2}\cos(t^2) \right\rangle$ with respect to its arc length measured from $t = 1$ in the direction of increasing t.

Solution:
We first compute the arc length function. We start with the derivative

$$\mathbf{r}'(t) = \langle 3t\cos(t^2), 4t, -3\sin(t^2)\rangle,$$

We now need its magnitude,

$$|\mathbf{r}'(t)| = \sqrt{9t^2\cos^2(t^2) + 16t^2 + 9\sin^2(t^2)},$$

$$|\mathbf{r}'(t)| = \sqrt{9t^2 + 16t^2} = \left(\sqrt{9+16}\right) t \quad \Rightarrow \quad |\mathbf{r}'(t)| = 5t.$$

Section 13.3

Example

Reparametrize the curve $\mathbf{r}(t) = \left\langle \frac{3}{2}\sin(t^2), 2t^2, \frac{3}{2}\cos(t^2) \right\rangle$ with respect to its arc length measured from $t = 1$ in the direction of increasing t.

Solution: Recall: $|\mathbf{r}'(t)| = 5t$. The arc length function is

$$s(t) = \int_1^t 5\tau \, d\tau = \frac{5}{2}\left(\tau^2\Big|_1^t\right) = \frac{5}{2}(t^2 - 1).$$

Inverting this function for t^2, we obtain $t^2 = \frac{2}{5}s + 1$.
The reparametrization of $\mathbf{r}(t)$ is given by

$$\hat{\mathbf{r}}(s) = \left\langle \frac{3}{2}\sin\left(\frac{2}{5}s + 1\right), 2\left(\frac{2}{5}s + 1\right), \frac{3}{2}\cos\left(\frac{2}{5}s + 1\right) \right\rangle. \qquad \lhd$$

Section 14.2

Example

Find an equation for the level surface of $f(x,y,z) = z - x^2 - y^2$ containing the point $P_0 = (3,-1,1)$.

Solution: Any level surface is the set of points is space solution of

$$z - x^2 - y^2 = k, \qquad k \in \mathbb{R}.$$

To find k, we evaluate the equation above at P_0,

$$1 - 9 - 1 = k \quad \Rightarrow \quad k = -9.$$

We conclude that $z - x^2 - y^2 = -9$. $\triangleleft$

Section 14.2

Example

Compute the limit $\displaystyle\lim_{(x,y)\to(4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$.

Solution: Remark: $f(x,y) = \dfrac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$ is not defined at $(4,3)$.
Recall from Calculus I: Multiply by the conjugate.

$$f(x,y) = \left[\frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}\right] \left[\frac{\sqrt{x} + \sqrt{y+1}}{\sqrt{x} + \sqrt{y+1}}\right]$$

$$f(x,y) = \frac{\big[x - (y+1)\big]}{(x - y - 1)\big[\sqrt{x} + \sqrt{y+1}\big]} = \frac{1}{\sqrt{x} + \sqrt{y+1}}.$$

We conclude that, $\displaystyle\lim_{(x,y)\to(4,3)} f(x,y) = \frac{1}{4}$. $\triangleleft$

Section 14.2

Example

Compute the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{xy}$.

Solution: Remark: $f(x,y) = \dfrac{x^2+y^2}{xy}$ is not defined at $(0,0)$.

Recall: It is often easier to prove that a limit does not exist.

The two paths theorem. Paths $x=0$ or $y=0$ are not useful.
First path, $y=x$. Then

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2+x^2}{x^2} = \lim_{x\to 0} 2 = 2.$$

Second path, $y=2x$. Then

$$\lim_{x\to 0} f(x,2x) = \lim_{x\to 0} \frac{x^2+4x^2}{x(2x)} = \lim_{x\to 0} \frac{5}{2} = \frac{5}{2}.$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE. $\triangleleft$

Section 14.3

Example

Given a function z defined by the equation $xy + z^3x - 2yz = 0$, find $\partial_x z$ in terms of x, y and z.

Solution: We use implicit differentiation:

$$y + z^3 + 3z^2\, z_x\, x - 2y\, z_x = 0,$$

$$z_x\,(3xz^2 - 2y) = -y - z^3$$

$$z_x = -\frac{(y+z^3)}{(3xz^2-2y)}.$$

$\triangleleft$

Section 14.4

Example

Given $f(x, y) = e^{xy}$, and the coordinate transformation $x = r\cos(\theta)$ and $y = r\sin(\theta)$, find $\hat{f}_r$ and $\hat{f}_\theta$, where $\hat{f}(r, \theta) = f(x(r, \theta), y(r, \theta))$.

Solution: The chain rule implies

$$\hat{f}_r = f_x\, x_r + f_y\, y_r, \qquad \hat{f}_\theta = f_x\, x_\theta + f_y\, y_\theta.$$

$$\hat{f}_r = ye^{xy}\, x_r + xe^{xy}\, y_r \quad \Rightarrow \quad \hat{f}_r e^{xy}(y\, x_r + x\, y_r);$$

$$\hat{f}_r = e^{r^2 \sin(\theta)\cos(\theta)}\big[r\sin(\theta)\cos(\theta) + r\cos(\theta)\sin(\theta)\big].$$

$$\hat{f}_\theta = ye^{xy}\, x_\theta + xe^{xy}\, y_\theta \quad \Rightarrow \quad \hat{f}_\theta = e^{xy}(y\, x_\theta + x\, y_\theta).$$

$$\hat{f}_\theta = e^{r^2 \sin(\theta)\cos(\theta)}\big[-r^2\sin(\theta)\sin(\theta) + r^2\cos(\theta)\cos(\theta)\big]. \quad \triangleleft$$

Also: $\hat{f}_r = r\sin(2\theta)\, e^{r^2\sin(2\theta)/2}$, and $\hat{f}_\theta = r^2\cos(2\theta)\, e^{r^2\sin(2\theta)/2}$.

Section 14.5

Example

Find all directions $\mathbf{u}$ such that $(D_{\mathbf{u}} f)(1, 1) = 0$, where $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

Solution: Recall: The $D_{\mathbf{u}} f = 0$ iff $\mathbf{u} \perp \nabla f$. The gradient is

$$f_x = \frac{2x(x^2 + y^2)}{(x^2 + y^2)^2} - \frac{(x^2 - y^2)2x}{(x^2 + y^2)^2} \quad \Rightarrow \quad f_x = \frac{4xy^2}{(x^2 + y^2)^2}.$$

$$f_y = \frac{-2y(x^2 + y^2)}{(x^2 + y^2)^2} - \frac{(x^2 - y^2)2y}{(x^2 + y^2)^2} \quad \Rightarrow \quad f_y = \frac{-4yx^2}{(x^2 + y^2)^2}.$$

$$\nabla f = \frac{4xy}{(x^2 + y^2)^2}\,\langle y, -x\rangle. \quad \Rightarrow \quad (\nabla f)(1, 1) = \langle 1, -1\rangle.$$

Example

Find all directions $\mathbf{u}$ such that $(D_{\mathbf{u}} f)(1,1) = 0$, where $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

Solution: Recall: $(\nabla f)(1,1) = \langle 1, -1 \rangle$.

We find all $\mathbf{u}$, unit, such that $\mathbf{u} \perp \langle 1, -1 \rangle$.

$$\langle u_x, u_y \rangle \cdot \langle 1, -1 \rangle = 0 \quad \Rightarrow \quad u_x = u_y.$$

Since $\mathbf{u}$ is unit, then

$$u_x^2 + u_y^2 = 1 \quad \Rightarrow \quad 2u_x^2 = 1 \quad \Rightarrow \quad u_x = \pm \frac{1}{\sqrt{2}}.$$

We conclude: $\mathbf{u} = \pm \dfrac{1}{\sqrt{2}} \langle 1, 1 \rangle$. ◁

Example

Find the direction of most rapid increase of the function $f(x,y,z) = \ln(x^3 + y^3 - 1) - y + 3z$ at the point $P_0 = (1,1,4)$.

Solution: Recall: The direction of most rapid increase is the direction of the gradient vector at P_0.

$$\nabla f = \frac{3x^2}{x^3 + y^3 - 1}\,\mathbf{i} + \Big(\frac{3y^2}{x^3 + y^3 - 1} - 1 \Big)\,\mathbf{j} + 3\mathbf{k}.$$

The direction of maximum increase is

$$(\nabla f)(1,1,4) = \langle 3, 2, 3 \rangle$$

◁

Section 14.5

Example

(a) Find the direction in which $f(x, y) = x^2 e^{3y}$ decreases the most rapidly at $P_0 = (-1, 0)$, and also find the directional derivative of $f(x, y)$ at P_0 along that direction.

(b) Find the directional derivative of $f(x, y)$ above at the point P_0 in the direction given by $\mathbf{v} = \langle -1, 1 \rangle$.

Solution:

(a) The direction f decreases the most rapidly is given by $-\nabla f$,

$$\nabla f(x, y) = \langle 2xe^{3y}, 3x^2 e^{3y} \rangle \quad \Rightarrow \quad -\nabla f(-1, 0) = \langle 2, -3 \rangle.$$

The value of the directional derivative along this direction is, $-|\nabla f(-1, 0)| = -\sqrt{9 + 4} = -\sqrt{13}$.

(b) A unit vector along $\langle -1, 1 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$, then,

$$D_u f(-1, 0) = \nabla f(-1, 0) \cdot \mathbf{u} = \langle -2, 3 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = \frac{5}{\sqrt{2}}. \quad \lhd$$

Section 14.6

Example

Find the linear approximation of $f(x, y) = x \cos(\pi y/2) - y^2 e^x$ at the point $(0, -1)$, and use it to approximate $f(0.1, -0.9)$.

Solution: $f(x, y) = x \cos(\pi y/2) - y^2 e^x$, $f(0, -1) = -1$.

$$f_x(x, y) = \cos(\pi y/2) - y^2 e^x, \quad f_x(0, -1) = \cos(-\pi/2) - 1 = -1.$$

$$f_y(x, y) = -x \sin(\pi y/2) \frac{\pi}{2} - 2ye^x \quad f_y(0, -1) = 2,$$

Then, the linear approximation $L(x, y)$ is given by

$$L(x, y) = -(x - 0) + 2(y + 1) - 1, \quad \Rightarrow \quad L(x, y) = -x + 2y + 1.$$

The linear approximation of $f(0.1, -0.9)$ is $L(0.1, -0.9)$,

$$L(0.1, -0.9) = -0.1 + 2(0.1) - 1 = 0.1 - 1 = -0.9. \quad \lhd$$

Section 14.6

Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution:

(a) $L(x, y) = f_x(-3, 2)\,(x + 3) + f_y(-3, 2)\,(y - 2) + f(-3, 2)$.

Since $f_x(x, y) = 2\cos(2x + 3y)$ and $f_y(x, y) = 3\cos(2x + 3y)$,

$$f_x(-3, 2) = 2\cos(-6 + 6) = 2, \quad f_y(-3, 2) = 3\cos(-6 + 6) = 3,$$

$$f(-3, 2) = \sin(-6 + 6) + 1 = 1.$$

the linear approximation is $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

Section 14.6

Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution: Recall: $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

(b) We use L to find the a linear approximation to $f(-2.9, 2.1)$.

We need to compute $L(-2.9, 2.1)$.

$$L(-2.9, 2.1) = 2(-2.9 + 3) + 3(2.1 - 2) + 2$$

$$L(-2.9, 2.1) = 2(0.1) + 3(0.1) + 1 \quad \Rightarrow \quad L(-2.9, 2.1) = 1.5.$$

$\triangleleft$

Exact value is close to 1.479.

Section 14.7

Example

(a) Find all the critical points of $f(x,y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) $\nabla f(x,y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle$, then,

$$x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x-4) = 0.$$

There are two solutions, $x = 0 \Rightarrow y = 0$, and $x = 4 \Rightarrow y = 8$.

That is, there are two critical points, $(0,0)$ and $(4,8)$.

Section 14.7

Example

(a) Find all the critical points of $f(x,y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:

(b) Recalling $\nabla f(x,y) = \langle 12y - 6x^2, 12x - 6y \rangle$, we compute

$$f_{xx} = -12x, \quad f_{yy} = -6, \quad f_{xy} = 12.$$

$$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 144 \left(\frac{x}{2} - 1 \right),$$

Since $D(0,0) = -144 < 0$, the point $(0,0)$ is a saddle point of f.

Since $D(4,8) = 144(2-1) > 0$, and $f_{xx}(4,8) = (-12)4 < 0$, the point $(4,8)$ is a local maximum of f. $\triangleleft$

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

We start finding the critical points inside the triangular region.

$$\nabla f(x,y) = \left\langle y - 2, x - \frac{1}{2}y \right\rangle = \langle 0, 0 \rangle, \quad \Rightarrow \quad y = 2, \quad y = 2x.$$

The solution is $(1,2)$. This point is outside in the triangular region given by the problem, so there is no critical point inside the region.

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

We now find the candidates for absolute maximum and minimum on the borders of the triangular region. We first record the boundary vertices:

$$\begin{aligned} (0,0) \quad &\Rightarrow \quad f(0,0) = 2, \\ (1,0) \quad &\Rightarrow \quad f(1,0) = 0, \\ (0,2) \quad &\Rightarrow \quad f(0,2) = 1. \end{aligned}$$

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

- The horizontal side of the triangle, $y = 0$, $x \in (0,1)$. Since

$$g(x) = f(x,0) = 2 - 2x, \quad \Rightarrow \quad g'(x) = -2 \neq 0.$$

there are no candidates in this part of the boundary.

- The vertical side of the triangle is $x = 0$, $y \in (0,2)$. Then,

$$g(y) = f(0,y) = 2 - \frac{1}{4}y^2, \quad \Rightarrow \quad g'(y) = -\frac{1}{2}y = 0,$$

so $y = 0$ and we recover the point $(0,0)$.

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

- The hypotenuse of the triangle $y = 2 - 2x$, $x \in (0,1)$. Then,

$$\begin{aligned} g(x) = f(x, 2-2x) &= 2 + x(2-2x) - 2x - \frac{1}{4}(2-2x)^2, \\ &= 2 + 2x - 2x^2 - 2x - (x^2 - 2x + 1), \\ &= 1 + 2x - 3x^2. \end{aligned}$$

Then, $g'(x) = 2 - 6x = 0$ implies $x = \frac{1}{3}$, hence $y = \frac{4}{3}$. The candidate is $\left(\frac{1}{3}, \frac{4}{3}\right)$.

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x, y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0, 0)$, $(1, 0)$, and $(0, 2)$.

Solution:

- Recall that we have obtained a candidate point $\left(\frac{1}{3}, \frac{4}{3}\right)$. We evaluate f at this point,

$$f\left(\frac{1}{3}, \frac{4}{3}\right) = 2 + \frac{4}{9} - \frac{2}{3} - \frac{1}{4}\frac{16}{9} = \frac{4}{3}.$$

Recalling that $f(0, 0) = 2$, $f(1, 0) = 0$, and $f(0, 2) = 1$, the absolute maximum is at $(0, 0)$, and the minimum is at $(1, 0)$. $\triangleleft$