## Review for the Final Exam.

- Monday, December 12, 10:00am - 12:00 noon. (2 hours.)
- Places:
  - Sctns 001, 002, 003, 015 in E-100 VMC (Vet. Medical Ctr.),
  - Sctns 004, 005, 016, 018 in S-105 SKH (South Kedzie Hall);
  - Sctns 017, in B-119 WH (Wells Halls).
- Chapters 12-16.
- Problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

## Review for Final Exam.

- Chapter 16, Sections 16.1-16.8.
- Chapter 15, Sections 15.1-15.5, 15.7.
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1-13.3.
- Chapter 12, Sections 12.1-12.6.

## Remark on Chapter 16.

Remark: The normal (flux) form of Green's Theorem is a two-dimensional restriction of the Divergence Theorem.

- The Divergence Theorem: $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D (\nabla\cdot\mathbf{F})\,dv.$
- Normal form of Green's Thrm: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S (\nabla\cdot\mathbf{F})\,dA.$

Remark: The tangential (circulation) form of Green's Theorem is a particular case of the Stokes Theorem when $C$, $S$ are flat ($z = 0$).

- The Stokes Theorem: $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma.$
- Tang. form of Green's Thrm: $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA.$

## Chapter 16, Integration in vector fields.

Example

Use the Divergence Theorem to find the flux of $\mathbf{F} = \langle xy^2, x^2y, y\rangle$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$, and $z = 1$.

Solution: Recall: $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D (\nabla\cdot\mathbf{F})\,dv.$ We start with

$$\nabla\cdot\mathbf{F} = \partial_x(xy^2) + \partial_y(x^2y) + \partial_z(y) \quad\Rightarrow\quad \nabla\cdot\mathbf{F} = y^2 + x^2.$$

The integration region is $D = \{x^2 + y^2 \leqslant 1,\ z\in[-1,1]\}$. So,

$$I = \iiint_D (\nabla\cdot\mathbf{F})\,dv = \iiint_D (x^2+y^2)\,dx\,dy\,dz.$$

We use cylindrical coordinates,

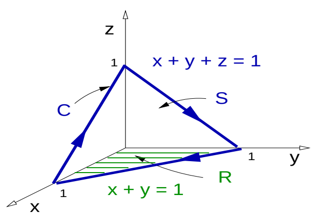
$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\,dz\,r\,dr\,d\theta = 2\pi\Big[\int_0^1 r^3\,dr\Big]\,(2) = 4\pi\Big(\frac{r^4}{4}\Big|_0^1\Big).$$

We conclude that $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \pi.$ $\triangleleft$

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path $C$ given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:




Recall: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$.

The surface $S$ is the level surface $f = 0$ of

$$f = x + y + z - 1$$

therefore, $\nabla f = \langle 1, 1, 1 \rangle$, $|\nabla f| = \sqrt{3}$ and $|\nabla f \cdot \mathbf{k}| = 1$.

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle, \qquad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy = \sqrt{3}\, dx\, dy.$$



Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path $C$ given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution: $\mathbf{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$ and $d\sigma = \sqrt{3}\, dx\, dy$.

We now compute the curl of $\mathbf{F}$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & xy & yz \end{vmatrix} = \langle (z - 0), -(0 - 2x), (y - 0) \rangle$$

so $\nabla \times \mathbf{F} = \langle z, 2x, y \rangle$. Therefore,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R \Big( \langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \Big) \sqrt{3}\, dx\, dy$$

## Chapter 16, Integration in vector fields.

### Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path $C$ given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:

$$I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R \Big( \langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \Big) \sqrt{3}\, dx\, dy.$$

$$I = \iint_R (z + 2x + y)\, dx\, dy, \qquad z = 1 - x - y,$$

$$I = \int_0^1 \int_0^{1-x} (1+x)\, dy\, dx = \int_0^1 (1+x)(1-x)\, dx = \int_0^1 (1-x^2)\, dx.$$

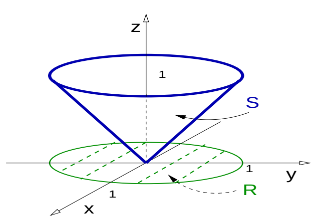
$$I = x\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \quad \Rightarrow \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3}.$$

## Chapter 16, Integration in vector fields.

### Example

Find the area of the cone $S$ given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through $S$.

Solution:




Recall: $A(S) = \iint_S d\sigma$. The surface $S$ is the level surface $f = 0$ of the function $f = x^2 + y^2 - z^2$. Also recall that

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy.$$

Since $\nabla f = 2\langle x, y, -z \rangle$, we get that

$$|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}, \quad z^2 = x^2 + y^2 \quad \Rightarrow \quad |\nabla f| = 2\sqrt{2}\, z.$$

Also $|\nabla f \cdot \mathbf{k}| = 2z$, therefore, $d\sigma = \sqrt{2}\, dx\, dy$, and then we obtain

$$A(S) = \iint_R \sqrt{2}\, dx\, dy = \int_0^{2\pi} \int_0^1 \sqrt{2} r\, dr\, d\theta = 2\pi\sqrt{2} \frac{r^2}{2}\Big|_0^1 = \sqrt{2}\, \pi.$$

## Chapter 16, Integration in vector fields.

Example

Find the area of the cone $S$ given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through $S$.

Solution: We now compute the outward flux $I = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$.

Since

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}\, z} \langle x, y, -z \rangle.$$

$$I = \iint_R \frac{1}{\sqrt{2}\, z} (x^2 + y^2) \sqrt{2} \, dx \, dy = \iint_R \sqrt{x^2 + y^2} \, dx \, dy.$$

Using polar coordinates, we obtain

$$I = \int_0^{2\pi} \int_0^1 r \, r \, dr \, d\theta = 2\pi \frac{r^3}{3} \Big|_0^1 \quad \Rightarrow \quad I = \frac{2\pi}{3}.$$

## Review for Final Exam.

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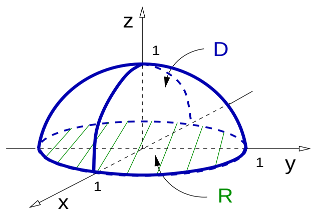
- **Chapter 15, Sections 15.1-15.5, 15.7.**
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1-13.3.
- Chapter 12, Sections 12.1-12.6.

## Chapter 15, Multiple integrals.

### Example

Find the volume of the region bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

### Solution:




So, $D = \{x^2 + y^2 \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2 - y^2\}$, and $R = \{x^2 + y^2 \leqslant 1,\ z = 0\}$. We know that

$$V(D) = \iiint_D dv = \iint_R \int_0^{1-x^2-y^2} dz\, dx\, dy.$$

Using cylindrical coordinates $(r, \theta, z)$, we get

$$V(D) = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} dz\, r\, dr\, d\theta = 2\pi \int_0^1 (1 - r^2)\, r\, dr.$$

Substituting $u = 1 - r^2$, so $du = -2r\, dr$, we obtain

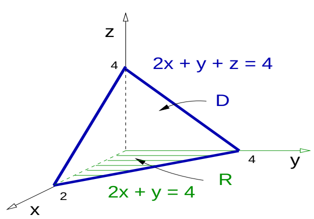
$$V(D) = 2\pi \int_1^0 u\, \frac{(-du)}{2} = \pi \int_0^1 u\, du = \pi \frac{u^2}{2}\Big|_0^1 \quad \Rightarrow \quad V(D) = \frac{\pi}{2}.$$

## Chapter 15, Multiple integrals.

### Example

Set up the integrals needed to compute the average of the function $f(x, y, z) = z \sin(x)$ on the bounded region $D$ in the first octant bounded by the plane $z = 4 - 2x - y$. Do not evaluate the integrals.

Solution: Recall: $\overline{f} = \dfrac{1}{V(D)} \iiint_D f\, dv.$




Since $V(D) = \displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx$,

we conclude that

$$\overline{f} = \frac{\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z \sin(x)\, dz\, dy\, dx}{\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx}.$$

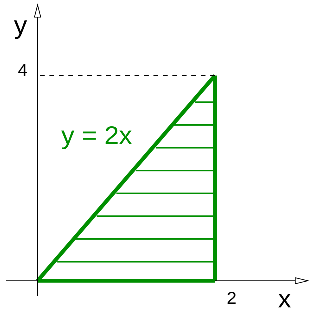
# Chapter 15, Multiple integrals.

## Example

Reverse the order of integration and evaluate the double integral

$$I = \int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy.$$

Solution: We see that $y \in [0, 4]$ and $x \in [0, y/2]$, that is,




Therefore, reversing the integration order means

$$I = \int_0^2 \int_0^{2x} e^{x^2}\, dy\, dx.$$

This integral is simple to compute,

$$I = \int_0^2 e^{x^2}\, x\, dx, \qquad u = x^2, \quad du = 2x\, dx,$$

$$I = \int_0^4 e^u\, du \quad \Rightarrow \quad I = e^4 - 1.$$