Review for the Final Exam.

- Monday, December 12, 10:00am - 12:00 noon. (2 hours.)
- Places:
  - Sctns 001, 002, 003, 015 in E-100 VMC (Vet. Medical Ctr.),
  - Sctns 004, 005, 016, 018 in S-105 SKH (South Kedzie Hall);
  - Sctns 017, in B-119 WH (Wells Halls).
- Chapters 12-16.
- Problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Final Exam.

- Chapter 16, Sections 16.1-16.8.
- Chapter 15, Sections 15.1-15.5, 15.7.
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1-13.3.
- Chapter 12, Sections 12.1-12.6.
Remark on Chapter 16.

Remark: The normal (flux) form of Green’s Theorem is a two-dimensional restriction of the Divergence Theorem.

▶ The Divergence Theorem: \[ \iiint_D (\nabla \cdot \mathbf{F}) \, dv. \]
▶ Normal form of Green’s Thrm: \[ \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S (\nabla \cdot \mathbf{F}) \, dA. \]

Remark: The tangential (circulation) form of Green’s Theorem is a particular case of the Stokes Theorem when \( C, S \) are flat \( (z = 0) \).

▶ The Stokes Theorem: \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma. \]
▶ Tang. form of Green’s Thrm: \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA. \]

Chapter 16, Integration in vector fields.

Example

Use the Divergence Theorem to find the flux of \( \mathbf{F} = (xy^2, x^2y, y) \) outward through the surface of the region enclosed by the cylinder \( x^2 + y^2 = 1 \) and the planes \( z = -1, \) and \( z = 1. \)

Solution: Recall: \[ \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) \, dv. \] We start with

\[ \nabla \cdot \mathbf{F} = \partial_x(xy^2) + \partial_y(x^2y) + \partial_z(y) \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = y^2 + x^2. \]

The integration region is \( D = \{ x^2 + y^2 \leq 1, \, z \in [-1, 1] \} \). So,

\[ I = \iiint_D (\nabla \cdot \mathbf{F}) \, dv = \iiint_D (x^2 + y^2) \, dx \, dy \, dz. \]

We use cylindrical coordinates,

\[ I = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 \, dz \, r \, dr \, d\theta = 2\pi \left[ \int_0^1 r^3 \, dr \right] (2) = 4\pi \left( \frac{r^4}{4} \Big|_0^1 \right). \]

We conclude that \[ \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \pi. \]
Example

Use Stokes' Theorem to find the work done by the force \( \mathbf{F} = \langle 2xz, xy, yz \rangle \) along the path \( \mathbf{C} \) given by the intersection of the plane \( x + y + z = 1 \) with the first octant, counterclockwise when viewed from above.

**Solution:**

Recall: \( \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma. \)

The surface \( S \) is the level surface \( f = 0 \) of 

\[ f = x + y + z - 1 \]

therefore, \( \nabla f = \langle 1, 1, 1 \rangle, \) \( |\nabla f| = \sqrt{3} \) and \( |\nabla f \cdot \mathbf{k}| = 1. \)

\[
\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle, \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy = \sqrt{3} \, dx \, dy.
\]

We now compute the curl of \( \mathbf{F}, \)

\[
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & xy & yz \end{vmatrix} = \langle z - 0, -(0 - 2x), (y - 0) \rangle
\]

so \( \nabla \times \mathbf{F} = \langle z, 2x, y \rangle. \) Therefore,

\[
\iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \iint_{R} \left( \langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) \sqrt{3} \, dx \, dy
\]
Example
Use Stokes' Theorem to find the work done by the force
\( \mathbf{F} = \langle 2xz, xy, yz \rangle \) along the path \( C \) given by the intersection of the plane \( x + y + z = 1 \) with the first octant, counterclockwise when viewed from above.

Solution:
\[
I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \iint_R \left( \langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) \sqrt{3} \, dx \, dy.
\]
\[
I = \int_0^1 \int_0^{1-x} (z + 2x + y) \, dx \, dy,
\]
\[
z = 1 - x - y,
\]
\[
I = \int_0^1 \int_0^{1-x} (1+x) \, dy \, dx = \int_0^1 (1+x)(1-x) \, dx = \int_0^1 (1-x^2) \, dx.
\]
\[
I = x \left|_0^1 - \frac{x^3}{3} \right|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3}.
\]

Example
Find the area of the cone \( S \) given by \( z = \sqrt{x^2 + y^2} \) for \( z \in [0,1] \). Also find the flux of the field \( \mathbf{F} = \langle x, y, 0 \rangle \) outward through \( S \).

Solution:
Recall: \( A(S) = \iint_S d\sigma \). The surface \( S \) is the level surface \( f = 0 \) of the function \( f = x^2 + y^2 - z^2 \). Also recall that
\[
d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy.
\]
Since \( \nabla f = 2\langle x, y, -z \rangle \), we get that
\[
|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}, \quad z^2 = x^2 + y^2 \Rightarrow |\nabla f| = 2\sqrt{2}z.
\]
Also \( |\nabla f \cdot \mathbf{k}| = 2z \), therefore, \( d\sigma = \sqrt{2} \, dx \, dy \), and then we obtain
\[
A(S) = \iint_R \sqrt{2} \, dx \, dy = \int_0^{2\pi} \int_0^1 \sqrt{2} r \, dr \, d\theta = 2\pi \sqrt{2} \left| \frac{1}{2} \right|_0^1 = \sqrt{2}\pi.
\]
Chapter 16, Integration in vector fields.

Example
Find the area of the cone $S$ given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through $S$.

Solution: We now compute the outward flux $I = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$.

Since

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}z} \langle x, y, -z \rangle.$$  

$$I = \iint_S \frac{1}{\sqrt{2}z} (x^2 + y^2)^{\frac{1}{2}} \, dx \, dy = \iint_S \sqrt{x^2 + y^2} \, dx \, dy.$$

Using polar coordinates, we obtain

$$I = \int_0^{2\pi} \int_0^1 r \, r \, dr \, d\theta = 2\pi \left[ \frac{r^3}{3} \right]^1_0 \Rightarrow I = \frac{2\pi}{3}.$$  

Review for Final Exam.

► Chapter 16, Sections 16.1-16.8.
► **Chapter 15, Sections 15.1-15.5, 15.7.**
► Chapter 14, Sections 14.1-14.7.
► Chapter 13, Sections 13.1-13.3.
► Chapter 12, Sections 12.1-12.6.
Chapter 15, Multiple integrals.

Example

Find the volume of the region bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:

So, $D = \{x^2 + y^2 \leq 1, \ 0 \leq z \leq 1 - x^2 - y^2\}$, and $R = \{x^2 + y^2 \leq 1, \ z = 0\}$. We know that

$$V(D) = \iiint_D dv = \iiint_R \int_0^{1-x^2-y^2} dz \ dx \ dy.$$  

Using cylindrical coordinates $(r, \theta, z)$, we get

$$V(D) = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} dz \ r \ dr \ d\theta = 2\pi \int_0^1 (1 - r^2) \ r \ dr.$$  

Substituting $u = 1 - r^2$, so $du = -2r \ dr$, we obtain

$$V(D) = 2\pi \int_0^1 \left(-\frac{du}{2}\right) = \pi \int_0^1 u \ du = \pi \frac{u^2}{2}\bigg|_0^1 \Rightarrow V(D) = \frac{\pi}{2}.$$  

Chapter 15, Multiple integrals.

Example

Set up the integrals needed to compute the average of the function $f(x, y, z) = z \sin(x)$ on the bounded region $D$ in the first octant bounded by the plane $z = 4 - 2x - y$. Do not evaluate the integrals.

Solution: Recall: $\bar{f} = \frac{1}{V(D)} \iiint_D f \ dv$.

Since $V(D) = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz \ dy \ dx$, we conclude that

$$\bar{f} = \frac{\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z \sin(x) \ dz \ dy \ dx}{\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz \ dy \ dx}.$$  

Example
Reverse the order of integration and evaluate the double integral
\[ I = \int_0^4 \int_{y/2}^2 e^{x^2} \, dx \, dy. \]

Solution: We see that \( y \in [0, 4] \) and \( x \in [0, y/2] \), that is,
\[ y = 2x \]
Therefore, reversing the integration order means
\[ I = \int_0^2 \int_0^{2x} e^{x^2} \, dy \, dx. \]
This integral is simple to compute,
\[ I = \int_0^2 e^{x^2} \, dx, \quad u = x^2, \quad du = 2x \, dx, \]
\[ I = \int_0^4 e^u \, du \quad \Rightarrow \quad I = e^4 - 1. \]