Review for Exam 4

- Sections 16.1-16.7.
- 50 minutes.
- 5 to 10 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4

- **(16.7) Stokes' Theorem.**
- (16.6) Surface integrals.
- (16.5) Surface area.
- (16.4) The Green Theorem in a plane.
- (16.3) Conservative fields, potential functions.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.1) Line integrals.

The Stokes Theorem (16.7)

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
$S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

Solution: Recall: $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

The surface S is the cylinder walls and its cover at $z = h$.
Therefore, the curve C is the circle $x^2 + y^2 = a^2$ at $z = 0$.
That circle can be parametrized (counterclockwise) as
$\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$.

The Stokes Theorem (16.7)

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
$S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

Solution: $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$. Hence

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \big(a^2\sin^2(t) + a^2\cos^2(t)\big)\, dt = \int_0^{2\pi} a^2\, dt.$$

We conclude that $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 2\pi a^2$.

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- (16.7) Stokes' Theorem.
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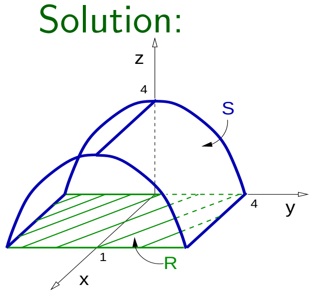
Surface integrals (16.6): Scalar fields

Example

Integrate the function $g(x, y, z) = x(16 - 4z)/y$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution:



We must compute: $I = \iint_S g \, d\sigma$.

Recall $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dx\, dy$, with $\mathbf{k} \perp R$

and in this case $f(x, y, z) = y^2 + 4z - 16$.

$$\nabla f = \langle 0, 2y, 4 \rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}.$$

Since $R = [0, 1] \times [-4, 4]$, its normal vector is $\mathbf{k}$ and $|\nabla f \cdot \mathbf{k}| = 4$. Then,

$$\iint_S g \, d\sigma = \iint_R \frac{x}{y} \left[16 - 4z(x, y)\right] \frac{2\sqrt{4 + y^2}}{4} \, dx\, dy.$$

Surface integrals (16.6)

Example

Integrate the function $g(x, y, z) = x(16 - 4z)/y$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution: $I = \iint_S g\, d\sigma = \iint_R \frac{x}{y}\left[16 - 4z(x, y)\right] \frac{2\sqrt{4 + y^2}}{4}\, dx\, dy.$

$$\frac{1}{2}\iint_R \frac{x}{y}\,(16 - 16 + y^2)\,\sqrt{4 + y^2}\, dx\, dy = \frac{1}{2}\iint_R xy\,\sqrt{4 + y^2}\, dx\, dy$$

$$I = \frac{1}{2}\Big[\int_{-4}^{4} y\sqrt{4 + y^2}\, dy\Big]\Big[\int_0^1 x\, dx\Big]$$

The first factor vanishes: An even function times an odd function is an odd function, which is integrated in $[-a, a]$.

We conclude that $\iint_S g\, d\sigma = 0$. $\triangleleft$

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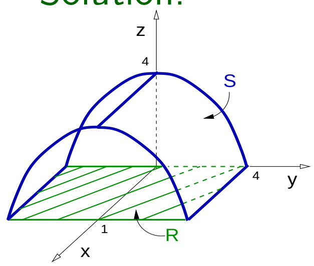
Surface area (16.5)

Example

Set up the integral for the area of the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$, $z = 0$.

Solution:



We must compute: $A(S) = \iint_S d\sigma$.

Recall $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dx\, dy$, with $\mathbf{k} \perp R$.

Recall: $f(x, y, z) = y^2 + 4z - 16$.

$$\nabla f = \langle 0, 2y, 4\rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}.$$

Since $R = [0, 1] \times [-4, 4]$, its normal vector is $\mathbf{k}$ and $|\nabla f \cdot \mathbf{k}| = 4$. Then,

$$A(S) = \iint_R \frac{2\sqrt{4 + y^2}}{4}\, dx\, dy \Rightarrow A(S) = \int_0^1 \int_{-4}^4 \frac{2\sqrt{4 + y^2}}{4}\, dy\, dx.$$

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The Green Theorem in a plane (16.4)

Example

Use the Green Theorem in the plane to find the flux of $\mathbf{F} = (x - y^2)\,\mathbf{i} + (x^2 + y)\,\mathbf{j}$ through the ellipse $9x^2 + 4y^2 = 36$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S \operatorname{div} \mathbf{F}\, dx\, dy.$

Recall: $\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y$. Here is simpler to compute the right-hand side than the left-hand side. $\operatorname{div} \mathbf{F} = 1 + 1 = 2$. Green's Theorem implies

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (2)\, dx\, dy. = 2\, A(R).$$

Since R is the ellipse $x^2/4 + y^2/9 = 1$, its area is $A(R) = (2)(3)\pi$. We conclude

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 12\pi. \qquad \triangleleft$$

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Conservative fields, potential functions (16.3)

Example

Compute $I = \int_{(0,0,0)}^{(1,-1,0)} 2x\cos(z)\,dx + z\,dy + (y - x^2\sin(z))\,dz.$

Solution: The integral is specified by the path end points. That suggests that the vector field is a gradient field.

$$\mathbf{F} = \langle 2x\cos(z), z, [y - x^2\sin(z)]\rangle = \nabla f = \langle \partial_x f, \partial_y f, \partial_z f\rangle.$$

$$\partial_x f = 2x\cos(z) \quad \Rightarrow \quad f = x^2\cos(z) + g(y,z).$$

$$\partial_y f = z = \partial_y g \;\Rightarrow\; g = yz + h(z) \;\Rightarrow\; f = x^2\cos(z) + yz + h(z).$$

$$\partial_z f = y - x^2\sin(z) = -x^2\sin(z) + y + h' \quad \Rightarrow \quad h' = 0$$

Since $f = x^2\cos(z) + yz + c$, we obtain

$$I = \int_{(0,0,0)}^{(1,-1,0)} \nabla f \cdot d\mathbf{r} = f(1,-1,0) - f(0,0,0) \quad \Rightarrow \quad I = 1. \triangleleft$$

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Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz\rangle$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: The flow (also called circulation) of the field $\mathbf{F}$ along a curve C parametrized by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt.$$

We use $t = x$ as the parameter of the curve $\mathbf{r}$, so we obtain

$$\mathbf{r}(t) = \langle t, t^2, t\rangle, \quad t \in [0,1] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, 2t, 1\rangle.$$

$$\mathbf{F}(t) = \langle t(t^2), (t^2)^2, -t^2(t)\rangle \quad \Rightarrow \quad \mathbf{F}(t) = \langle t^3, t^4, -t^3\rangle.$$

Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz\rangle$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: $\mathbf{r}'(t) = \langle 1, 2t, 1\rangle$ for $t \in [0,1]$ and $\mathbf{F}(t) = \langle t^3, t^4, -t^3\rangle$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \langle t^3, t^4, -t^3\rangle \cdot \langle 1, 2t, 1\rangle\, dt,$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^3 + 2t^5 - t^3)\, dt = \int_0^1 2t^5\, dt = \frac{2}{6}\, t^6\Big|_0^1.$$

We conclude that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}.$ ◁

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- **(16.1) Line integrals.**

Line integrals (16.1)

Example

Integrate the function $f(x,y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0,2]$, from the point $(0,0)$ to $(2,2)$.

Solution: We have to compute $I = \int_C f\,ds$, by that we mean

$$\int_C f\,ds = \int_{t_0}^{t_1} f\big(x(t), y(t)\big)\,|\mathbf{r}'(t)|\,dt,$$

where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \in [t_0, t_1]$ is a parametrization of the path C. In this case the path is given by the parabola $y = x^2/2$, so a simple parametrization is to use $x = t$, that is,

$$\mathbf{r}(t) = \Big\langle t, \frac{t^2}{2}\Big\rangle, \quad t \in [0,2] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, t\rangle.$$

Line integrals (16.1)

Example

Integrate the function $f(x,y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0,2]$, from the point $(0,0)$ to $(2,2)$.

Solution: $\mathbf{r}(t) = \left\langle t, \frac{t^2}{2} \right\rangle$ for $t \in [0,2]$, and $\mathbf{r}'(t) = \langle 1, t \rangle$.

$$\int_C f\,ds = \int_{t_0}^{t_1} f\big(x(t), y(t)\big)\,|\mathbf{r}'(t)|\,dt = \int_0^2 \frac{t^3}{t^2/2}\sqrt{1+t^2}\,dt,$$

$$\int_C f\,ds = \int_0^2 2t\sqrt{1+t^2}\,dt, \quad u = 1+t^2, \quad du = 2t\,dt.$$

$$\int_C f\,ds = \int_1^5 u^{1/2}\,du = \frac{2}{3}\,u^{3/2}\Big|_1^5 = \frac{2}{3}\big(5^{3/2}-1\big).$$

We conclude that $\int_C f\,ds = \frac{2}{3}\big(5\sqrt{5}-1\big)$. $\triangleleft$