Review for Exam 4

- Sections 16.1-16.6.
- 50 minutes.
- 5 to 10 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4

- **(16.6) Surface integrals.**
- (16.5) Surface area.
- (16.4) The Green Theorem in a plane.
- (16.3) Conservative fields, potential functions.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.1) Line integrals.
Surface integrals (16.6): Scalar fields

**Example**

Integrate the function \( g(x, y, z) = x\sqrt{4 + y^2} \) over the surface cut from the parabolic cylinder \( z = 4 - y^2/4 \) by the planes \( x = 0 \), \( x = 1 \) and \( z = 0 \).

**Solution:**

We must compute: \( I = \iint_S g \, d\sigma \).

Recall \( d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} \, dx \, dy \), with \( k \perp R \) and in this case \( f(x, y, z) = y^2 + 4z - 16 \).

\[ \nabla f = \langle 0, 2y, 4 \rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}. \]

Since \( R = [0, 1] \times [-4, 4] \), its normal vector is \( k \) and \( |\nabla f \cdot k| = 4 \).

Then,

\[ \iint_S g \, d\sigma = \iint_R (x\sqrt{4 + y^2}) \frac{2\sqrt{4 + y^2}}{4} \, dx \, dy. \]

Surface integrals (16.6)

**Example**

Integrate the function \( g(x, y, z) = x\sqrt{4 + y^2} \) over the surface cut from the parabolic cylinder \( z = 4 - y^2/4 \) by the planes \( x = 0 \), \( x = 1 \) and \( z = 0 \).

**Solution:**

\[ \iint_S g \, d\sigma = \iint_R (x\sqrt{4 + y^2}) \frac{2\sqrt{4 + y^2}}{4} \, dx \, dy. \]

\[ \iint_S g \, d\sigma = \frac{1}{2} \iint_R (4 + y^2) \, dx \, dy = \frac{1}{2} \int_{-4}^{4} \int_{0}^{1} (4 + y^2) \, dx \, dy \]

\[ \iint_S g \, d\sigma = \frac{1}{2} \left[ \int_{-4}^{4} (4 + y^2) \, dy \right] \left[ \int_{0}^{1} x \, dx \right] = \frac{1}{2} \left( 4y + \frac{y^3}{3} \right) \bigg|_{-4}^{4} \left( \frac{x^2}{2} \right) \bigg|_{0}^{1} \]

\[ \iint_S g \, d\sigma = \frac{1}{2} \left( 8\frac{4}{3} \right) \frac{1}{2} = 8 \left( 1 + \frac{4}{3} \right) \quad \Rightarrow \quad \iint_S g \, d\sigma = \frac{56}{3}. \]
Surface area (16.5)

Example

Set up the integral for the area of the surface cut from the parabolic cylinder \( z = 4 - y^2/4 \) by the planes \( x = 0, \, x = 1, \) and \( z = 0. \)

Solution:

We must compute: \( A(S) = \iint_S d\sigma. \)

Recall \( d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} \, dx \, dy, \) with \( k \perp R. \)

Recall: \( f(x, y, z) = y^2 + 4z - 16. \)

\[
\nabla f = \langle 0, 2y, 4 \rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}.
\]

Since \( R = [0, 1] \times [-4, 4], \) its normal vector is \( k \) and \( |\nabla f \cdot k| = 4. \)

Then,

\[
A(S) = \iint_R \frac{2\sqrt{4 + y^2}}{4} \, dx \, dy \Rightarrow A(S) = \int_0^1 \int_{-4}^4 \frac{2\sqrt{4 + y^2}}{4} dy \, dx.
\]

\( \bigtriangleup \)
Review for Exam 4

- (16.6) Surface integrals.
- (16.5) Surface area.
- **(16.4) The Green Theorem in a plane.**
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### The Green Theorem in a plane (16.4)

**Example**

Use the Green Theorem in the plane to find the flux of $\mathbf{F} = (x - y^2)\mathbf{i} + (x^2 + y)\mathbf{j}$ through the ellipse $9x^2 + 4y^2 = 36$.

**Solution:** Recall: $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S \text{div} \, \mathbf{F} \, dx \, dy$.

Recall: $\text{div} \, \mathbf{F} = \partial_x F_x + \partial_y F_y$. Here is simpler to compute the right-hand side than the left-hand side. $\text{div} \, \mathbf{F} = 1 + 1 = 2$.

Green's Theorem implies

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (2) \, dx \, dy = 2 \, A(R).$$

Since $R$ is the ellipse $x^2/4 + y^2/9 = 1$, its area is $A(R) = (2)(3)\pi$. We conclude

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 12\pi. \quad \triangle$$
Conservative fields, potential functions (16.3)

Example

Compute \( I = \int_{(0,0,0)}^{(1,-1,0)} 2x \cos(z) \, dx + z \, dy + (y - x^2 \sin(z)) \, dz \).

Solution: The integral is specified by the path end points. That suggests that the vector field is a gradient field.

\[ F = \langle 2x \cos(z), z, [y - x^2 \sin(z)] \rangle = \nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle. \]

\[ \partial_x f = 2x \cos(z) \quad \Rightarrow \quad f = x^2 \cos(z) + g(y, z). \]

\[ \partial_y f = z = \partial_y g \quad \Rightarrow \quad g = yz + h(z) \quad \Rightarrow \quad f = x^2 \cos(z) + yz + h(z). \]

\[ \partial_z f = y - x^2 \sin(z) = -x^2 \sin(z) + y + h' \quad \Rightarrow \quad h' = 0 \]

Since \( f = x^2 \cos(z) + yz + c \), we obtain

\[ I = \int_{(0,0,0)}^{(1,-1,0)} \nabla f \cdot dr = f(1,-1,0) - f(0,0,0) \quad \Rightarrow \quad I = 1. \]
Example

Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz \rangle$ from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: The flow (also called circulation) of the field $\mathbf{F}$ along a curve $C$ parametrized by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt.$$ 

We use $t = x$ as the parameter of the curve $\mathbf{r}$, so we obtain

$$\mathbf{r}(t) = \langle t, t^2, t \rangle, \quad t \in [0, 1] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, 2t, 1 \rangle.$$  

$$\mathbf{F}(t) = \langle t(t^2), (t^2)^2, -t^2(t) \rangle \quad \Rightarrow \quad \mathbf{F}(t) = \langle t^3, t^4, -t^3 \rangle.$$
Vector fields, work, circulation, flux (plane) (16.2)

Example

Find the flow of the velocity field \( \mathbf{F} = \langle xy, y^2, -yz \rangle \) from the point \((0, 0, 0)\) to the point \((1, 1, 1)\) along the curve of intersection of the cylinder \(y = x^2\) with the plane \(z = x\).

Solution: \( \mathbf{r}'(t) = \langle 1, 2t, 1 \rangle \) for \( t \in [0, 1] \) and \( \mathbf{F}(t) = \langle t^3, t^4, -t^3 \rangle \).

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt = \int_{0}^{1} \langle t^3, t^4, -t^3 \rangle \cdot \langle 1, 2t, 1 \rangle \, dt,
\]

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{1} (t^3 + 2t^5 - t^3) \, dt = \int_{0}^{1} 2t^5 \, dt = \frac{2}{6} t^6 \bigg|_{0}^{1}.
\]

We conclude that \( \int_{C} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3} \). △

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Line integrals (16.1)

Example
Integrate the function \( f(x, y) = x^3/y \) along the plane curve \( C \) given by \( y = x^2/2 \) for \( x \in [0, 2] \), from the point \((0, 0)\) to \((2, 2)\).

Solution: We have to compute 
\[
\int_C f \, ds = \int_{t_0}^{t_1} f(x(t), y(t)) |r'(t)| \, dt,
\]
where \( r(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) is a parametrization of the path \( C \). In this case the path is given by the parabola \( y = x^2/2 \), so a simple parametrization is to use \( x = t \), that is,
\[
r(t) = \langle t, \frac{t^2}{2} \rangle, \quad t \in [0, 2] \quad \Rightarrow \quad r'(t) = \langle 1, t \rangle.
\]

\[
\int_C f \, ds = \int_{2}^{0} t^3 \sqrt{1 + t^2} \, dt = \int_{1}^{5} u^{3/2} \, du = \frac{2}{3} \left( 5^{3/2} - 1 \right).
\]
We conclude that \( \int_C f \, ds = \frac{2}{3} (5\sqrt{5} - 1) \). \( \triangleq \)