The divergence of a vector field in space

**Definition**
The *divergence* of a vector field \( \mathbf{F} = \langle F_x, F_y, F_z \rangle \) is the scalar field

\[
\text{div} \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.
\]

**Remarks:**
- It is also used the notation \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} \).
- The divergence of a vector field measures the expansion (positive divergence) or contraction (negative divergence) of the vector field.
- A heated gas expands, so the divergence of its velocity field is positive.
- A cooled gas contracts, so the divergence of its velocity field is negative.
The divergence of a vector field in space

Example
Find the divergence and the curl of \( \mathbf{F} = \langle 2xyz, -xy, -z^2 \rangle \).

Solution: Recall: \( \text{div } \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z \).

\[
\partial_x F_x = 2yz, \quad \partial_y F_y = -x, \quad \partial_z F_z = -2z.
\]

Therefore \( \nabla \cdot \mathbf{F} = 2yz - x - 2z \), that is \( \nabla \cdot \mathbf{F} = 2z(y - 1) - x \).

Recall: \( \text{curl } \mathbf{F} = \nabla \times \mathbf{F} \).

\[
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & -xy & -z^2 \end{vmatrix} = \langle (0 - 0), -(0 - 2xy), (-y - 2xz) \rangle
\]

We conclude: \( \nabla \times \mathbf{F} = \langle 0, 2xy, -(2xz + y) \rangle \).

\[\triangle]\n
The divergence of a vector field in space

Example
Find the divergence of \( \mathbf{F} = \frac{\mathbf{r}}{\rho^3} \), where \( \mathbf{r} = \langle x, y, z \rangle \), and \( \rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \). (Notice: \( |\mathbf{F}| = 1/\rho^2 \).)

Solution: The field components are \( F_x = \frac{x}{\rho^3} \), \( F_y = \frac{y}{\rho^3} \), \( F_z = \frac{z}{\rho^3} \).

\[
\partial_x F_x = \partial_x \left[ x(x^2 + y^2 + z^2)^{-3/2} \right] = \frac{1}{\rho^3} - 3 \frac{x^2}{\rho^5} \Rightarrow \partial_y F_y = \frac{1}{\rho^3} - 3 \frac{y^2}{\rho^5}, \quad \partial_z F_z = \frac{1}{\rho^3} - 3 \frac{z^2}{\rho^5}.
\]

\[
\nabla \cdot \mathbf{F} = \frac{3}{\rho^3} - 3 \frac{(x^2 + y^2 + z^2)}{\rho^5} = \frac{3}{\rho^3} - 3 \frac{\rho^2}{\rho^5} = \frac{3}{\rho^3} - 3 \frac{3}{\rho^3}.
\]

We conclude: \( \nabla \cdot \mathbf{F} = 0 \). \[\triangle\]
The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.

The Divergence Theorem in space.

- The meaning of Curls and Divergences.

Applications in electromagnetism:
  - Gauss’ law. (Divergence Theorem.)
  - Faraday’s law. (Stokes Theorem.)

The Divergence Theorem in space

Theorem

The flux of a differentiable vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ across a closed oriented surface $S \subset \mathbb{R}^3$ in the direction of the surface outward unit normal vector $\mathbf{n}$ satisfies the equation

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV,$$

where $V \subset \mathbb{R}^3$ is the region enclosed by the surface $S$.

Remarks:

- The volume integral of the divergence of a field $\mathbf{F}$ in a volume $V$ in space equals the outward flux (normal flow) of $\mathbf{F}$ across the boundary $S$ of $V$.

- The expansion part of the field $\mathbf{F}$ in $V$ minus the contraction part of the field $\mathbf{F}$ in $V$ equals the net normal flow of $\mathbf{F}$ across $S$ out of the region $V$. 
Example
Verify the Divergence Theorem for the field \( F = \langle x, y, z \rangle \) over the sphere \( x^2 + y^2 + z^2 = R^2 \).

Solution: Recall: \( \iiint_S F \cdot n \, d\sigma = \iiint_V (\nabla \cdot F) \, dV \).

We start with the flux integral across \( S \). The surface \( S \) is the level surface \( f = 0 \) of the function \( f(x, y, z) = x^2 + y^2 + z^2 - R^2 \). Its outward unit normal vector \( n \) is

\[
 n = \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \langle 2x, 2y, 2z \rangle, \quad |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2R,
\]

We conclude that \( n = \frac{1}{R} \langle x, y, z \rangle \), where \( z = z(x, y) \).

Since \( d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} \, dx \, dy \), then \( d\sigma = \frac{R}{z} \, dx \, dy \), with \( z = z(x, y) \).

The integral on the sphere \( S \) can be written as the sum of the integral on the upper half plus the integral on the lower half, both integrated on the disk \( R = \{x^2 + y^2 \leq R^2, \ z = 0\} \), that is,

\[
 \iint_S F \cdot n \, d\sigma = 2R \iint_R \frac{R}{z} \, dx \, dy.
\]
The Divergence Theorem in space

Example
Verify the Divergence Theorem for the field \( \mathbf{F} = \langle x, y, z \rangle \) over the sphere \( x^2 + y^2 + z^2 = R^2 \).

Solution: \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 2R \iint_R \frac{R}{z} \, dx \, dy \).
Using polar coordinates on \( \{z = 0\} \), we get
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 2 \int_0^{2\pi} \int_0^R \frac{R^2}{\sqrt{R^2 - r^2}} r \, dr \, d\theta.
\]
The substitution \( u = R^2 - r^2 \) implies \( du = -2r \, dr \), so,
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi R^2 \int_0^{R^2} u^{-1/2} \left( \frac{-du}{2} \right) = 2\pi R^2 \int_0^{R^2} u^{-1/2} \, du
\]
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 2\pi R^2 \left( 2u^{1/2} \right) \bigg|_0^{R^2} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi R^3.
\]

The Divergence Theorem in space

Example
Verify the Divergence Theorem for the field \( \mathbf{F} = \langle x, y, z \rangle \) over the sphere \( x^2 + y^2 + z^2 = R^2 \).

Solution: \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi R^3 \).

We now compute the volume integral \( \iiint_V \nabla \cdot \mathbf{F} \, dV \). The divergence of \( \mathbf{F} \) is \( \nabla \cdot \mathbf{F} = 1 + 1 + 1 \), that is, \( \nabla \cdot \mathbf{F} = 3 \). Therefore
\[
\iiint_V \nabla \cdot \mathbf{F} \, dV = 3 \iiint_V \, dV = 3 \left( \frac{4}{3} \pi R^3 \right)
\]
We obtain \( \iiint_V \nabla \cdot \mathbf{F} \, dV = 4\pi R^3 \).
We have verified the Divergence Theorem in this case. \( \triangleleft \)
The Divergence Theorem in space

Example
Find the flux of the field \( \mathbf{F} = \frac{\mathbf{r}}{\rho^3} \) across the boundary of the region between the spheres of radius \( R_1 > R_0 > 0 \), where \( \mathbf{r} = \langle x, y, z \rangle \), and \( \rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \).

Solution: We use the Divergence Theorem

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV.
\]

Since \( \nabla \cdot \mathbf{F} = 0 \), then \( \iiint_V (\nabla \cdot \mathbf{F}) \, dV = 0 \). Therefore

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0.
\]

The flux along any surface \( S \) vanishes as long as \( \mathbf{0} \) is not included in the region surrounded by \( S \). (\( \mathbf{F} \) is not differentiable at \( \mathbf{0} \).) △

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- **The meaning of Curls and Divergences.**
- Applications in electromagnetism:
  - Gauss’ law. (Divergence Theorem.)
  - Faraday’s law. (Stokes Theorem.)
The meaning of Curls and Divergences

Remarks: The meaning of the Curl and the Divergence of a vector field $\mathbf{F}$ is best given through the Stokes and Divergence Theorems.

$\nabla \times \mathbf{F} = \lim_{S \to \{P\}} \frac{1}{A(S)} \oint_C \mathbf{F} \cdot d\mathbf{r},$

where $S$ is a surface containing the point $P$ with boundary given by the loop $C$ and $A(S)$ is the area of that surface.

$\nabla \cdot \mathbf{F} = \lim_{R \to \{P\}} \frac{1}{V(R)} \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma,$

where $R$ is a region in space containing the point $P$ with boundary given by the closed orientable surface $S$ and $V(R)$ is the volume of that region.

The Divergence Theorem. (Sect. 16.8)

The divergence of a vector field in space.

The Divergence Theorem in space.

The meaning of Curls and Divergences.

Applications in electromagnetism:

- Gauss’ law. (Divergence Theorem.)
- Faraday’s law. (Stokes Theorem.)
Applications in electromagnetism: Gauss’ Law

**Gauss’ law:** Let \( q : \mathbb{R}^3 \to \mathbb{R} \) be the charge density in space, and \( E : \mathbb{R}^3 \to \mathbb{R}^3 \) be the electric field generated by that charge. Then

\[
\iiint_R q \, dV = k \iint_S E \cdot n \, d\sigma,
\]

that is, the total charge in a region \( R \) in space with closed orientable surface \( S \) is proportional to the integral of the electric field \( E \) on this surface \( S \).

The Divergence Theorem relates surface integrals with volume integrals, that is,

\[
\iint_S E \cdot n \, d\sigma = \iiint_R (\nabla \cdot E) \, dV.
\]

Using the Divergence Theorem we obtain the differential form of Gauss’ law,

\[
\nabla \cdot E = \frac{1}{k} q.
\]

Applications in electromagnetism: Faraday’s Law

**Faraday’s law:** Let \( B : \mathbb{R}^3 \to \mathbb{R}^3 \) be the magnetic field across an orientable surface \( S \) with boundary given by the loop \( C \), and let \( E : \mathbb{R}^3 \to \mathbb{R}^3 \) measured on that loop. Then

\[
\frac{d}{dt} \iint_S B \cdot n \, d\sigma = -\oint_C E \cdot dr,
\]

that is, the time variation of the magnetic flux across \( S \) is the negative of the electromotive force on the loop.

The Stokes Theorem relates line integrals with surface integrals, that is,

\[
\oint_C E \cdot r = \iint_S (\nabla \times E) \, d\sigma.
\]

Using the Divergence Theorem we obtain the differential form of Gauss’ law,

\[
\partial_t B = -\nabla \times E.
\]