## The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- The curl of conservative fields.
- Stokes' Theorem in space.
- Idea of the proof of Stokes' Theorem.

## The curl of a vector field in space

**Definition**
The *curl* of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ in $\mathbb{R}^3$ is the vector field

$$\operatorname{curl} \mathbf{F} = (\partial_2 F_3 - \partial_3 F_2)\,\mathbf{i} + (\partial_3 F_1 - \partial_1 F_3)\,\mathbf{j} + (\partial_1 F_2 - \partial_2 F_1)\,\mathbf{k}.$$

Remark: Since the following formula holds,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

then one also uses the notation

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}.$$

Remark: The curl of a vector field measures the rotational component of the vector field at every point of its domain.

In 3-dimensions a vector is needed to collect this information.

## The curl of a vector field in space

Example

Find the curl of the vector field $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$.

Solution: Since $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$, we get,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xz & xyz & -y^2 \end{vmatrix} =$$

$$\big(\partial_y(-y^2) - \partial_z(xyz)\big)\,\mathbf{i} - \big(\partial_x(-y^2) - \partial_z(xz)\big)\,\mathbf{j} + \big(\partial_x(xyz) - \partial_y(xz)\big)\,\mathbf{k},$$

$$= \big(-2y - xy\big)\,\mathbf{i} - \big(0 - x\big)\,\mathbf{j} + \big(yz - 0\big)\,\mathbf{k},$$

We conclude that

$$\nabla \times \mathbf{F} = \langle -y(2 + x), x, yz \rangle.$$

$\triangleleft$

## The Stokes Theorem. (Sect. 16.7)

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- **The curl of conservative fields.**
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## The curl of conservative fields

Recall: A vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ is *conservative* iff there exists a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

Theorem

*If a vector field* $\mathbf{F}$ *is conservative, then* $\nabla \times \mathbf{F} = \mathbf{0}$.

Remark:

- This Theorem is usually written as $\nabla \times (\nabla f) = \mathbf{0}$.
- The converse is true only on simple connected sets. That is, if a vector field $\mathbf{F}$ satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ on a simple connected domain $D$, then there exists a scalar field $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

Proof of the Theorem:

$$\nabla \times \mathbf{F} = \Big\langle \big(\partial_y \partial_z f - \partial_z \partial_y f\big), -\big(\partial_x \partial_z f - \partial_z \partial_x f\big), \big(\partial_x \partial_y f - \partial_y \partial_x f\big) \Big\rangle$$

□

## The curl of conservative fields

Example

Is the vector field $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$ conservative?

Solution: We have shown that $\nabla \times \mathbf{F} = \langle -y(2+x), x, yz \rangle$.
Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is not conservative. ◁

Example

Is the vector field $\mathbf{F} = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$ conservative in $\mathbb{R}^3$?

Solution: Notice that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix}$$

$$= \big\langle (6xyz^2 - 6xyz^2), -(3y^2z^2 - 3y^2z^2), (2yz^3 - 2yz^3) \big\rangle = \mathbf{0}.$$

Since $\nabla \times \mathbf{F} = \mathbf{0}$ and $\mathbb{R}^3$ is simple connected, then $\mathbf{F}$ is conservative, that is, there exists $f$ in $\mathbb{R}^3$ such that $\mathbf{F} = \nabla f$. ◁

# The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- The curl of conservative fields.
- **Stokes' Theorem in space.**
- Idea of the proof of Stokes' Theorem.

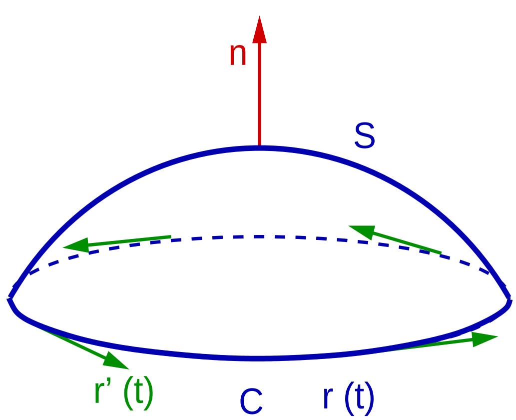
# Stokes' Theorem in space

Theorem

*The circulation of a differentiable vector field* $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ *around the boundary* $C$ *of the oriented surface* $S \subset D$ *satisfies the equation*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma,$$

*where* $d\mathbf{r}$ *points counterclockwise when the unit vector* $\mathbf{n}$ *normal to* $S$ *points in the direction to the viewer (right-hand rule).*

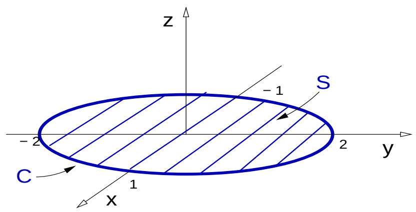


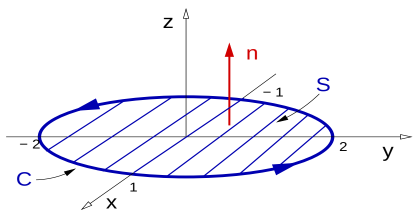
## Stokes' Theorem in space

### Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: We compute both sides in $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$.




We start computing the circulation integral on the ellipse $x^2 + \frac{y^2}{2^2} = 1$.

If we choose the upward normal to $S$, we have to choose a counterclockwise parametrization for $C$.

So we choose, for $t \in [0, 2\pi]$,

$$\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle.$$

and the right-hand rule normal $\mathbf{n}$ to $S$
$\mathbf{n} = \langle 0, 0, 1 \rangle$.

## Stokes' Theorem in space

### Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: Recall: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$, with
$\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle$, $t \in [0, 2\pi]$ and $\mathbf{n} = \langle 0, 0, 1 \rangle$.
The circulation integral is:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^{2\pi} \langle \cos^2(t), 2\cos(t), 0 \rangle \cdot \langle -\sin(t), 2\cos(t), 0 \rangle \, dt.$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left[ -\cos^2(t)\sin(t) + 4\cos^2(t) \right] dt.$$

## Stokes' Theorem in space

### Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \big[-\cos^2(t)\sin(t) + 4\cos^2(t)\big]\, dt.$

The substitution on the first term $u = \cos(t)$ and $du = -\sin(t)\, dt$, implies $\displaystyle\int_0^{2\pi} -\cos^2(t)\sin(t)\, dt = \int_1^1 u^2\, du = 0.$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 4\cos^2(t)\, dt = \int_0^{2\pi} 2\big[1 + \cos(2t)\big]\, dt.$$

Since $\displaystyle\int_0^{2\pi} \cos(2t)\, dt = 0$, we conclude that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi.$

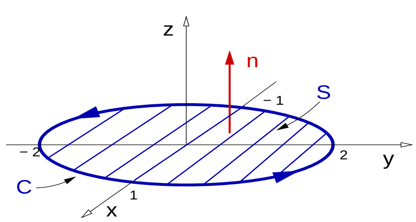
## Stokes' Theorem in space

### Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$ and $\mathbf{n} = \langle 0, 0, 1 \rangle$.

We now compute the right-hand side in Stokes' Theorem.




$$I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2x & z^2 \end{vmatrix} \quad \Rightarrow \quad \nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle.$$

$S$ is the flat surface $\{x^2 + \frac{y^2}{2^2} \leqslant 1, \ z = 0\}$, so $d\sigma = dx\, dy$.

## Stokes' Theorem in space

**Example**

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: $\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$, $\mathbf{n} = \langle 0, 0, 1 \rangle$, $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, and $d\sigma = dx\, dy$.

Then, $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \langle 0, 0, 2 \rangle \cdot \langle 0, 0, 1 \rangle\, dy\, dx.$

The right-hand side above is twice the area of the ellipse. Since we know that an ellipse $x^2/a^2 + y^2/b^2 = 1$ has area $\pi ab$, we obtain

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 4\pi.$$

This verifies Stokes' Theorem. $\triangleleft$

## Stokes' Theorem in space

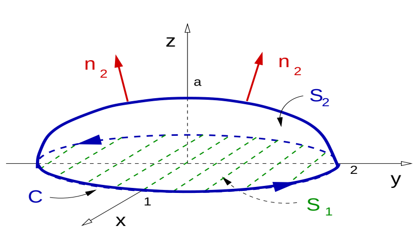
Remark: Stokes' Theorem implies that for any smooth field $\mathbf{F}$ and any two surfaces $S_1$, $S_2$ having the same boundary curve $C$ holds,

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1\, d\sigma_1 = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, d\sigma_2.$$

**Example**

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \ : \ x^2 + \frac{y^2}{2^2} + \frac{z^2}{a^2} = 1, \ z \geqslant 0\}$.

Solution: (The previous example was the case $a \to 0$.)




We must verify Stokes' Theorem on $S_2$,

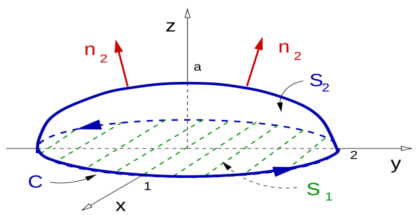
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, d\sigma_2.$$

## Stokes' Theorem in space

### Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \ : \ x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{a^2} = 1, \ z \geqslant 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$, $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, $I = \displaystyle\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2$.




$S_2$ is the level surface $\mathbb{F} = 0$ of

$$\mathbb{F}(x, y, z) = x^2 + \frac{y^2}{2^2} + \frac{z^2}{a^2} - 1.$$

$$\mathbf{n}_2 = \frac{\nabla\mathbb{F}}{|\nabla\mathbb{F}|}, \quad \nabla\mathbb{F} = \left\langle 2x, \frac{y}{2}, \frac{2z}{a^2} \right\rangle, \quad (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 = 2\,\frac{2z/a^2}{|\nabla\mathbb{F}|}.$$

$$d\sigma_2 = \frac{|\nabla\mathbb{F}|}{|\nabla\mathbb{F} \cdot \mathbf{k}|} = \frac{|\nabla\mathbb{F}|}{2z/a^2} \quad \Rightarrow \quad (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = 2.$$

## Stokes' Theorem in space

### Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \ : \ x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{a^2} = 1, \ z \geqslant 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = 2$.

Therefore,

$$\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = \iint_{S_1} 2 \, dx \, dy = 2(2\pi).$$

We conclude that $\displaystyle\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = 4\pi$, no matter what is the value of $a > 0$. $\triangleleft$

## The Stokes Theorem. (Sect. 16.7)

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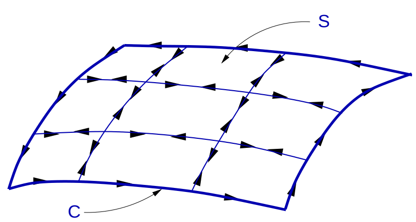
## Idea of the proof of Stokes' Theorem

Split the surface $S$ into $n$ surfaces $S_i$, for $i = 1, \cdots, n$, as it is done in the figure for $n = 9$.




$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \oint_{C_i} \mathbf{F} \cdot d\mathbf{r}_i$$

$$\simeq \sum_{i=1}^{n} \oint_{\tilde{C}_i} \mathbf{F} \cdot d\tilde{\mathbf{r}}_i \quad (\tilde{C}_i \text{ the border of small rectangles});$$

$$= \sum_{i=1}^{n} \iint_{\tilde{R}_i} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_i \, dA \text{ (Green's Theorem on a plane)};$$

$$\simeq \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma. \qquad \square$$