Surface area and surface integrals. (Sect. 16.6)

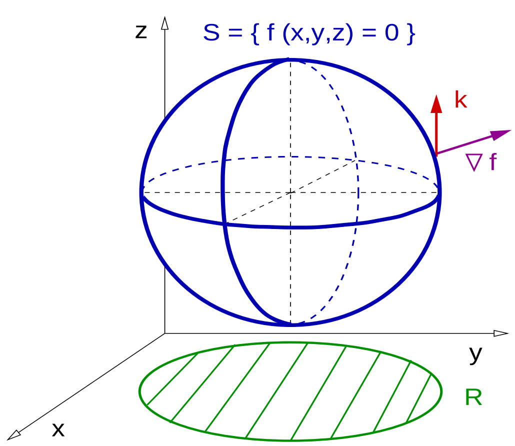
- Review: The area of a surface in space.
- Surface integrals of a scalar field.
- The flux of a vector field on a surface.
- Mass and center of mass thin shells.

Review: The area of a surface in space

Theorem

Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, the area of a level surface $S = \{f(x,y,z) = 0\}$, over a closed, bounded region R in the plane $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$



Remark: Eq. (7), page 949, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

The vector $\mathbf{p}$ in the textbook is the vector normal to R. In our case $\mathbf{p} = \mathbf{k}$.

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Surface integrals of a scalar field

Theorem

The integral of a continuous scalar function $g : \mathbb{R}^3 \to \mathbb{R}$ over a surface S defined as the level set of $f(x, y, z) = 0$ over the bounded plane R is given by

$$\iint_S g\, d\sigma = \iint_R g \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA,$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$.

Remark: In the particular case $g = 1$, we recover the formula for the area $A(S) = \iint_S d\sigma$ of the surface S, that is,

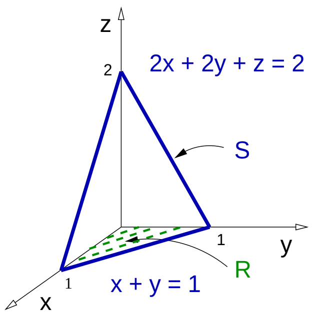
$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA.$$

Surface integrals of a scalar field

Example

Integrate the function $g(x, y, z) = x + y + z$ over the surface given by the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

Solution: Recall: $\displaystyle\iint_S g\, d\sigma = \iint_R g\, \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA.$



Here $f = 2x + 2y + z - 2$, so the surface S is given by $f = 0$ in the first octant. Hence, the region R is on the $z = 0$ plane, (therefore $\mathbf{p} = \mathbf{k}$) given by the triangle with sides $x = 0$, $y = 0$ and $x + y = 1$.

So, $\nabla f = \langle 2, 2, 1\rangle$, hence $|\nabla f| = 3$, and $|\nabla f \cdot \mathbf{k}| = 1$. Therefore

$$\iint_S g\, d\sigma = \iint_R g(x, y, z)\, 3\, dA.$$

Surface integrals of a scalar field

Example

Integrate the function $g(x, y, z) = x + y + z$ over the surface given by the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

Solution: Recall: $\displaystyle\iint_S g\, d\sigma = \iint_R g(x, y, z)\, 3\, dA.$

Now, function g must be evaluated on the surface S. That means

$$g(x, y, z(x, y)) = x + y + z(x, y) = x + y + (2 - 2x - 2y).$$

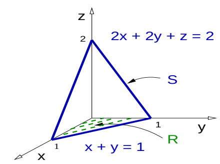
$$g(x, y, z(z, y)) = 2 - x - y.$$

$$\iint_S g\, d\sigma = 3 \iint_R (2 - x - y)\, dA.$$

Surface integrals of a scalar field

Example

Integrate the function $g(x,y,z) = x + y + z$ over the surface given by the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.



Solution: $\displaystyle\iint_S g\, d\sigma = 3 \iint_R (2 - x - y)\, dA.$

The region R is the triangle in the plane $z = 0$ given by the lines $x = 0$, $y = 0$, and $x + y = 1$. Therefore,

$$3\int_0^1 \int_0^{1-y} (2-x-y)\, dx\, dy = 3\int_0^1 \Big[(2-y)\Big(x\Big|_0^{1-y}\Big) - \Big(\frac{x^2}{2}\Big|_0^{1-y}\Big)\Big]\, dy$$

$$\iint_S g\, d\sigma = 3\int_0^1 \Big[(2-y)(1-y) - \frac{1}{2}(1-y)^2\Big]\, dy$$

$$\iint_S g\, d\sigma = 3\int_0^1 \Big(\frac{3}{2} - 2y + \frac{y^2}{2}\Big)\, dy \quad \Rightarrow \quad \iint_S g\, d\sigma = 2. \quad \triangleleft$$

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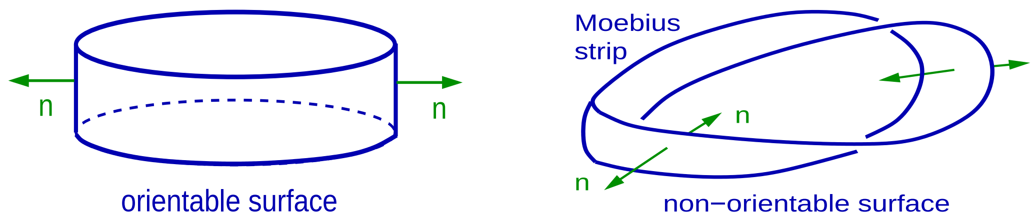
The flux of a vector field on a surface

Definition

A surface $S \subset \mathbb{R}^3$ is called *orientable* if it is possible to define on S a continuous, unit vector field $\mathbf{n}$ normal to S.



Definition

The *flux* of a continuous vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ over an orientable surface S in the direction of a unit normal $\mathbf{n}$ is given by

$$\mathbb{F} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Remark: $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA$, where S is the level surface $f = 0$.

The flux of a vector field on a surface

Example

Find the flux of the field $\mathbf{F} = \langle 0, 0, z \rangle$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

Solution: Recall: $\mathbb{F} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$.

In this case S is the level surface $f = 0$, for $f = x^2 + y^2 + z^2 - a^2$. The unit normal vector $\mathbf{n}$ is proportional to ∇f.

$$\nabla f = \langle 2x, 2y, 2z \rangle, \quad |\nabla f| = 2\sqrt{x^2 + y^2 + z^2}.$$

On the surface S we have that $x^2 + y^2 + z^2 = a^2$, therefore, $|\nabla f| = 2a$ on this surface. We obtain that on S the appropriate normal vector is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} \quad \Rightarrow \quad \mathbf{n} = \frac{1}{a}\langle x, y, z \rangle, \quad z|_S = z(x, y).$$

The flux of a vector field on a surface

Example

Find the flux of the field $\mathbf{F} = \langle 0, 0, z \rangle$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

Solution: Recall: $\mathbb{F} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ and $\mathbf{n} = \frac{1}{a}\langle x, y, z \rangle$ on S.

Since $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$, and $\nabla f = 2\langle x, y, z \rangle$, which on S says $|\nabla f| = 2a$, we conclude, $d\sigma = \dfrac{2a}{2z}\, dx\, dy$, hence $d\sigma = \dfrac{a}{z}\, dx\, dy$.

$$\mathbb{F} = \iint_R \Big(\langle 0, 0, z \rangle \cdot \frac{1}{a}\langle x, y, z \rangle \Big) \frac{a}{z}\, dx\, dy.$$

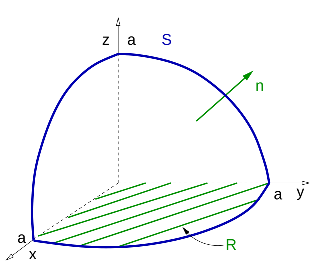
$$\mathbb{F} = \iint_R \frac{z^2}{a}\,\frac{a}{z}\, dx\, dy \quad \Rightarrow \quad \mathbb{F} = \iint_R z\, dx\, dy, \quad z|_S = z(x, y).$$

The flux of a vector field on a surface.

Example

Find the flux of the field $\mathbf{F} = \langle 0, 0, z \rangle$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

Solution: Recall: $\mathbb{F} = \iint_R z\, dx\, dy$, and z must be evaluated on S.



The integral is only on the first octant.

$$\mathbb{F} = \iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy.$$

We use polar coordinates on $R \subset \{z = 0\}$.

$$\mathbb{F} = \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta. \quad u = a^2 - r^2, \quad du = -2r\, dr.$$

$$\mathbb{F} = \frac{\pi}{2} \int_{a^2}^0 u^{1/2} \frac{(-du)}{2} = \frac{\pi}{4} \int_0^{a^2} u^{1/2}\, du = \frac{\pi}{4}\frac{2}{3}\, (a^2)^{3/2} \Rightarrow \mathbb{F} = \frac{\pi a^3}{6}.$$

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Mass and center of mass of thin shells

Definition

The *mass* M of a thin shell described by the surface S in space with mass per unit area function $\rho : S \to \mathbb{R}$ is given by

$$M = \iint_S \rho \, d\sigma.$$

The *center of mass* $\bar{\mathbf{r}} = \langle \overline{x}_1, \overline{x}_2, \overline{x}_3 \rangle$ of the thin shell above is

$$\overline{x}_i = \frac{1}{M} \iint_S x_i \, \rho \, d\sigma, \qquad i = 1, 2, 3.$$

Remark:

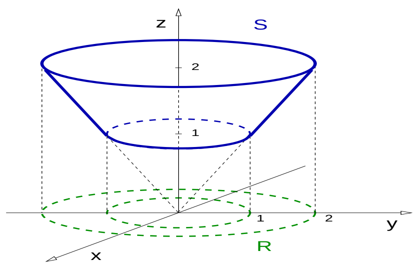
- The *centroid vector* is the particular case of the center of mass vector for an object with constant density.
- See in the textbook the definitions of moments of inertia I_{x_i}, with $i = 1, 2, 3$, for thin shells.

Mass and center of mass of thin shells

Example

Find the centroid of the surface S given by $x^2 + y^2 = z^2$ between the planes $z = 1$ and $z = 2$.

Solution: The surface S is a cone section, given in the figure.



We first compute the area, M, of S,

$$M = \iint_S d\sigma = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.$$

Here $f = x^2 + y^2 - z^2$, therefore,

$$\nabla f = \langle 2x, 2y, -2z \rangle.$$

Hence $|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}$, evaluated on S. Since $z^2 = x^2 + y^2$, we get $|\nabla f| = 2\sqrt{2}\, z$. Also $\nabla f \cdot \mathbf{k} = -2z$. So,

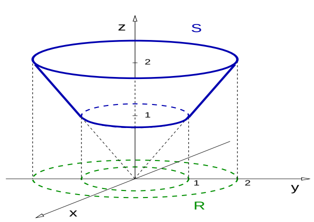
$$\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \frac{2\sqrt{2}\, z}{2z} = \sqrt{2} \quad \Rightarrow \quad M = \iint_R \sqrt{2}\, dA.$$

Mass and center of mass of thin shells

Example

Find the centroid of the surface S given by $x^2 + y^2 = z^2$ between the planes $z = 1$ and $z = 2$.

Solution: Recall: $\dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \sqrt{2}$ and $M = \iint_R \sqrt{2}\, dA$.



We use polar coordinates in $\{z = 0\}$,

$$M = \sqrt{2} \int_0^{2\pi} \int_1^2 r \, dr \, d\theta = 2\pi\sqrt{2} \Big(\frac{r^2}{2} \Big|_1^2 \Big)$$

We conclude $M = 3\sqrt{2}\, \pi$.

By symmetry, the only non-zero component of the centroid is $\bar{z}$.

$$\bar{z} = \frac{1}{M} \iint_R z \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA = \frac{\sqrt{2}}{3\sqrt{2}\pi} \iint_R \sqrt{x^2 + y^2} \, dx \, dy.$$

$$\bar{z} = \frac{1}{3\pi} \int_0^{2\pi} \int_1^2 r^2 dr \, d\theta = \frac{2\pi}{3\pi} \Big(\frac{r^3}{3} \Big|_1^3 \Big) = \frac{2}{9} (8 - 1) \quad \Rightarrow \quad \bar{z} = \frac{14}{9}.$$