Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- The area of a surface in space.
  - The surface is given in parametric form.
  - The surface is given in explicit form.

Review: Arc length and line integrals

- The integral of a function $f : [a, b] \to \mathbb{R}$ is
  $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\,\Delta x.$
- The arc length of a curve $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^3$ in space is
  $\displaystyle s_{t_1,t_0} = \int_{t_0}^{t_1} |\mathbf{r}'(t)|\,dt.$
- The integral of a function $f : \mathbb{R}^3 \to \mathbb{R}$ along a curve
  $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^3$ is $\displaystyle\int_C f\,ds = \int_{t_0}^{t_1} f\big(\mathbf{r}(t)\big)\,|\mathbf{r}'(t)|\,dt.$
- The circulation of a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ along a curve
  $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^3$ is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{u}\,ds = \int_{t_0}^{t_1} \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\,dt.$
- The flux of a function $\mathbf{F} : \{z = 0\} \cap \mathbb{R}^3 \to \{z = 0\} \cap \mathbb{R}^3$ along
  a loop $\mathbf{r} : [t_0, t_1] \to \{z = 0\} \cap \mathbb{R}^3$ is $\displaystyle\mathbb{F} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds.$

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Review: Double integral of a scalar function

- The double integral of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on a region $R \subset \mathbb{R}^2$, which is the volume under the graph of f and above the $z = 0$ plane, and is given by

$$\iint_R f \, dA = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \, \Delta x \, \Delta y.$$

- The area of a flat surface $R \subset \mathbb{R}^2$ is the particular case $f = 1$, that is, $A(R) = \iint_R dA$.

We will show how to compute:

- The area of a *non-flat surface* in space. (Today.)
- The integral of a scalar function f on a surface is space.
- The flux of a vector-valued function $\mathbf{F}$ on a surface in space.

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Explicit, implicit, parametric equations of surfaces

Review: Curves on $\mathbb{R}^2$ can be defined in:

- Explicit form, $y = f(x)$;
- Implicit form, $F(x, y) = 0$;
- Parametric form, $\mathbf{r}(t) = \langle x(t), y(t) \rangle$.

  The vector $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ is tangent to the curve.

Review: Surfaces in $\mathbb{R}^3$ can be defined in:

- Explicit form, $z = f(x, y)$;
- Implicit form, $F(x, y, z) = 0$;
- Parametric form, $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$.
  Two vectors tangent to the surface are

$$\partial_u \mathbf{r}(u, v) = \langle \partial_u x(u, v), \partial_u y(u, v), \partial_u z(u, v) \rangle,$$

$$\partial_v \mathbf{r}(u, v) = \langle \partial_v x(u, v), \partial_v y(u, v), \partial_v z(u, v) \rangle.$$

Explicit, implicit, parametric equations of surfaces

Example

Find a parametric expression for the cone $z = \sqrt{x^2 + y^2}$, and two tangent vectors.

Solution: Use cylindrical coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$. Parameters of the surface: $u = r$, $v = \theta$. Then

$$x(r,\theta) = r\cos(\theta), \quad y(r,\theta) = r\sin(\theta), \quad z(r,\theta) = r.$$

Using vector notation, a parametric equation of the cone is

$$\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r\rangle.$$

Two tangent vectors to the cone are $\partial_r \mathbf{r}$ and $\partial_\theta \mathbf{r}$,

$$\partial_r \mathbf{r} = \langle \cos(\theta), \sin(\theta), 1\rangle, \quad \partial_\theta \mathbf{r} = \langle -r\sin(\theta), r\cos(\theta), 0\rangle. \qquad \triangleleft$$

Explicit, implicit, parametric equations of surfaces

Example

Find a parametric expression for the sphere $x^2 + y^2 + z^2 = R^2$, and two tangent vectors.

Solution: Use spherical coordinates:
$x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$.
Parameters of the surface: $u = \theta$, $v = \phi$.

$$x = R\cos(\theta)\sin(\phi), \quad y = R\sin(\theta)\sin(\phi), \quad z = R\cos(\phi).$$

Using vector notation, a parametric equation of the cone is

$$\mathbf{r}(\theta,\phi) = R\,\langle \cos(\theta)\sin(\phi),\; \sin(\theta)\sin(\phi),\; \cos(\phi)\rangle.$$

Two tangent vectors to the paraboloid are $\partial_\theta \mathbf{r}$ and $\partial_\phi \mathbf{r}$,

$$\partial_\theta \mathbf{r} = R\,\langle -\sin(\theta)\sin(\phi),\; \cos(\theta)\sin(\phi),\; 0\rangle,$$

$$\partial_\phi \mathbf{r} = R\,\langle \cos(\theta)\cos(\phi),\; \sin(\theta)\cos(\phi),\; -\sin(\phi)\rangle. \qquad \triangleleft$$

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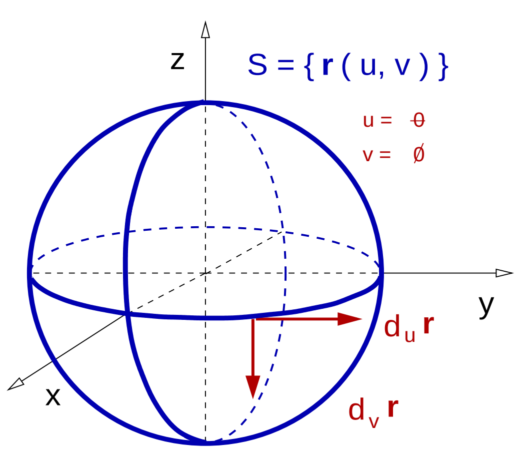
The area of a surface in parametric form

Theorem
Given a smooth surface S with parametric equation $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ *for* $u \in [u_0, u_1]$ *and* $v \in [v_0, v_1]$ *is given by*

$$A(S) = \int_{u_0}^{u_1} \int_{v_0}^{v_1} |\partial_u \mathbf{r} \times \partial_v \mathbf{r}|\, dv\, du.$$



Remark: The function

$$d\sigma = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}|\, dv\, du.$$

represents the area of a small region on the surface.

This is the generalization to surfaces of the arc-length formula for the length of a curve.

The area of a surface in parametric form

Example

Find an expression for the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: Use cylindrical coordinates. The surface in parametric form is

$$\mathbf{r}(r, \theta) = \langle r\cos(\theta),\ r\sin(\theta),\ r^2 \rangle.$$

The tangent vectors to the surface $\partial_r \mathbf{r}$, $\partial_\theta \mathbf{r}$ are

$$\partial_r \mathbf{r} = \langle \cos(\theta), \sin(\theta), 2r \rangle, \quad \partial_\theta \mathbf{r} = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle.$$

$$\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix}$$

$$\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle.$$

The area of a surface in parametric form

Example

Find an expression for the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: Recall: $\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle$.

$$|\partial_r \mathbf{r} \times \partial_\theta \mathbf{r}| = \sqrt{4r^4 + r^2} = r\sqrt{1 + 4r^2}.$$

$$A(S) = \int_0^{2\pi} \int_0^2 r\,\sqrt{1 + 4r^2}\,dr\,d\theta.$$

This integral will be done later on by substitution. The result is:

$$A(S) = \frac{\pi}{6}\left[(17)^{3/2} - 1\right].$$

$\triangleleft$

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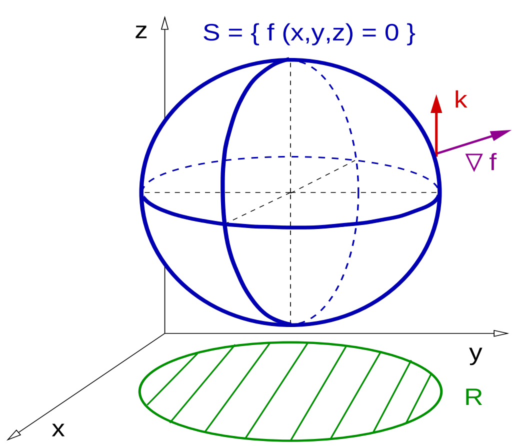
The area of a surface in space in explicit form

Theorem

Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, the area of a level surface $S = \{f(x, y, z) = 0\}$, over a closed, bounded region R in the plane $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$



Remark: Eq. (7), page 949, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

The vector $\mathbf{p}$ in the textbook is the vector normal to R. In our case $\mathbf{p} = \mathbf{k}$.

The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x,y,z) = 0\}$ over a flat region R in $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$

Example

Find the area of $S = \{z - 1 = 0\}$ over R in $\{z = 0\}$.

Solution: This is simple: $f(x,y,z) = z - 1$, so $\nabla f = \mathbf{k}$, hence

$$\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = 1 \quad \Rightarrow \quad A(S) = \iint_R dx\, dy = A(R).$$

◁

Remark: The formula for $A(S)$ is reasonable: Every flat horizontal surface S over a flat horizontal region R satisfies $A(S) = A(R)$.

The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x,y,z) = 0\}$ over a flat region R in $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$

Example

Find the area of $S = \{y + z - 1 = 0\}$ over R in $\{z = 0\}$.

Solution: The plane S intersects the horizontal plane at a $\pi/4$ angle. So, $f(x,y,z) = y + z - 1$, and $\nabla f = \mathbf{j} + \mathbf{k}$, hence

$$\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \sqrt{2} \Rightarrow A(S) = \iint_R \sqrt{2}\, dx\, dy \Rightarrow A(S) = \sqrt{2}\, A(R).$$

◁

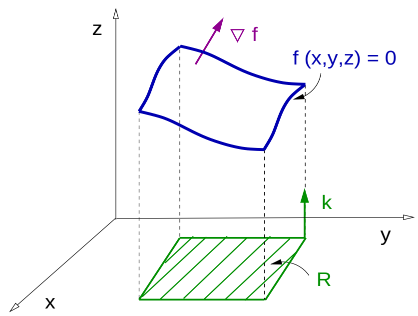
Remark: The formula for $A(S)$ is still reasonable: Every flat surface S having an angle $\pi/4$ over a flat horizontal region R satisfies $A(S) = \sqrt{2}\, A(R)$.

The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat horizontal region R in $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$

Remark: The formula for $A(S)$ can be interpreted as follows: The factor $\dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}$ is the angle correction function needed to obtain the $A(S)$ by correcting the $A(R)$ by the relative inclination of S with respect to R.

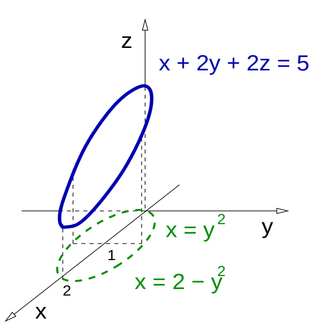


The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution:



The surface is given by $f = 0$ with

$$f(x, y, z) = x + 2y + 2z - 5.$$

The region R is in the plane $z = 0$,

$$R = \left\{ \begin{array}{c} (x, y, z) \ : \ z = 0, \ y \in [-1, 1] \\ x \in [y^2, (2 - y^2)] \end{array} \right\}.$$

Recall: $A(S) = \iint_R \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA$. Here $\nabla f = \langle 1, 2, 2 \rangle$.

The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA$. Here $\nabla f = \langle 1, 2, 2\rangle$.

Therefore: $|\nabla f| = \sqrt{1+4+4} = 3$, and $|\nabla f \cdot \mathbf{k}| = 2$.

And the region $R = \{(x, y) \ : \ y \in [-1, 1], \ x \in [y^2, (2 - y^2)]\}$.

So we can write down the expression for $A(S)$ as follows,

$$A(S) = \iint_R \frac{3}{2}\, dx\, dy = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx\, dy.$$

The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx\, dy$.

$$A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2)\, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2)\, dy$$

$$A(S) = 3 \int_{-1}^{1} (1 - y^2)\, dy = 3\Big(y - \frac{y^3}{3}\Big)\Big|_{-1}^{1} = 3\Big(1 - \frac{1}{3} + 1 - \frac{1}{3}\Big)$$

$$A(S) = 3\Big(2 - \frac{2}{3}\Big) = 3\,\frac{4}{3} \quad \Rightarrow \quad A(S) = 4. \qquad \triangleleft$$

The area of a surface in space in explicit form

Example

Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: The surface is the level surface of the function $f(x, y, z) = x^2 + y^2 - z$. The region R is the disk $z = x^2 + y^2 \leqslant 4$.

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy, \quad \nabla f = \langle 2x, 2y, -1 \rangle, \quad \nabla f \cdot \mathbf{k} = -1,$$

$$A(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy.$$

Since R is a disk radius 2, it is convenient to use polar coordinates in $\mathbb{R}^2$. We obtain

$$A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta.$$

The area of a surface in space in explicit form

Example

Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: Recall: $A(S) = \displaystyle\int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta.$

$$A(S) = 2\pi \int_0^2 \sqrt{1 + 4r^2}\, r\, dr, \qquad u = 1 + 4r^2,\ du = 8r\, dr.$$

$$A(S) = \frac{2\pi}{8} \int_1^{17} u^{1/2}\, du = \frac{2\pi}{8}\, \frac{2}{3} \left(u^{3/2} \Big|_1^{17} \right).$$

We conclude: $A(S) = \dfrac{\pi}{6} \big[(17)^{3/2} - 1 \big].$ $\triangleleft$

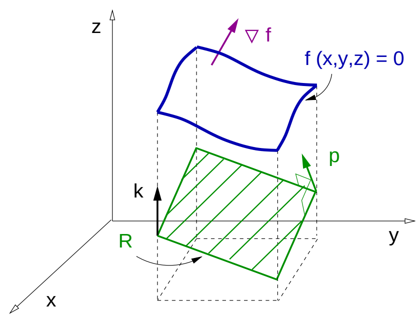
The area of a surface in space in explicit form

Remark: The formula for the area of a surface in space can be generalized as follows.

Theorem

The area of a surface S given by $f(x, y, z) = 0$ over a closed and bounded plane region R in space is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA,$$



where $\mathbf{p}$ is a unit vector normal to the region R and $\nabla f \cdot \mathbf{p} \neq 0$.

The area of a surface in space in explicit form

Proof in a simple case: Assume that the surface us given in explicit form:

$$S = \{(x, y, z) \;:\; z = g(x, y)\},$$

On the one hand, a simple parametric form is to use $u = x$, $v = y$ and $z(u, v) = g(u, v)$. Hence

$$\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle \quad \Rightarrow \quad \begin{cases} \partial_x \mathbf{r} = \langle 1, 0, \partial_x g\rangle \\ \partial_y \mathbf{r} = \langle 0, 1, \partial_y g\rangle, \end{cases}$$

$$\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1\rangle$$

On the other hand, an implicit form for the surface is

$$f(x, y, z) = g(x, y) - z$$

Therefore, $\quad \partial_x f = \partial_x g, \quad \partial_y f = \partial_y g, \quad \partial_z f = -1.$

The area of a surface in space in explicit form

Proof in a simple case: Recall: $\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1 \rangle$ and

$$\partial_x f = \partial_x g, \quad \partial_y f = \partial_y g, \quad \partial_z f = -1.$$

One can show (with chain rule) that $\partial_x \mathbf{r} \times \partial_y \mathbf{r}$ is given by

$$\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \Big\langle \frac{\partial_x f}{\partial_z f}, \frac{\partial_x f}{\partial_z f}, 1 \Big\rangle \quad \Rightarrow \quad \partial_x \mathbf{r} \times \partial_y \mathbf{r} = \frac{1}{\partial_z f} \langle \partial_x f, \partial_y f, \partial_z f \rangle.$$

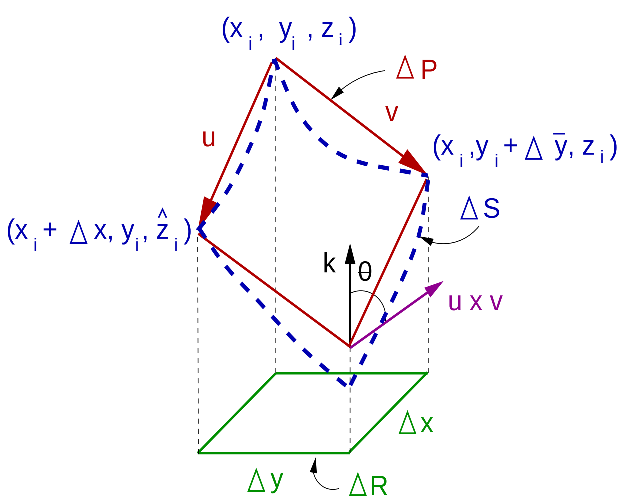
That is, $\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \dfrac{\nabla f}{\nabla f \cdot \mathbf{k}}$. We then obtain

$$A(S) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} |\partial_x \mathbf{r} \times \partial_y \mathbf{r}| \, dy \, dx = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.$$

$\square$

The area of a surface in space in explicit form

Proof: Introduce a partition in $R \subset \mathbb{R}^2$, and consider an arbitrary rectangle ΔR in that partition. We compute the area ΔP.



It is simple to see that

$$\Delta P = |\mathbf{u} \times \mathbf{v}|,$$

and

$$\mathbf{u} = \langle \Delta x, 0, (z_i - \hat{z}_i) \rangle,$$
$$\mathbf{v} = \langle 0, \Delta y, (z_i - \overline{z}_i) \rangle.$$

Therefore,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & (z_i - \hat{z}_i) \\ 0 & \Delta y & (z_i - \overline{z}_i) \end{vmatrix} = \langle -\Delta y (z_i - \hat{z}_i), -\Delta x (z_i - \overline{z}_i), \Delta x \Delta y \rangle.$$

The area of a surface in space in explicit form

Proof: Recall: $\mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle$.

The linearization of $f(x, y, z)$ at (x_i, y_i, z_i) implies

$$f(x, y, z) \simeq f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).$$

Since $f(x_i, y_i, z_i) = 0$, $f(x_i + \Delta x, y_i, \hat{z}_i) = 0$, $f(x_i, y_i + \Delta y, \bar{z}_i) = 0$,

$$0 = (\partial_x f)_i \Delta x + (\partial_z f)_i (z_i - \hat{z}_i) \quad \Rightarrow \quad (z_i - \hat{z}_i) = -\frac{(\partial_x f)_i}{(\partial_z f)_i} \Delta x,$$

$$0 = (\partial_y f)_i \Delta y + (\partial_z f)_i (z_i - \bar{z}_i) \quad \Rightarrow \quad (z_i - \bar{z}_i) = -\frac{(\partial_y f)_i}{(\partial_z f)_i} \Delta y.$$

$$\mathbf{u} \times \mathbf{v} = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \frac{\Delta x \Delta y}{(\partial_z f)_i} \Rightarrow \mathbf{u} \times \mathbf{v} = \frac{(\nabla f)_i}{(\nabla f \cdot \mathbf{k})_i} \Delta x \Delta y.$$

$$\Delta P = \frac{|(\nabla f)_i|}{|(\nabla f \cdot \mathbf{k})_i|} \Delta x \Delta y \quad \Rightarrow \quad A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA. \quad \square$$