Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- The area of a surface in space.
  - The surface is given in parametric form.
  - The surface is given in explicit form.

Review: Arc length and line integrals

- The integral of a function \( f : [a, b] \to \mathbb{R} \) is
  \[
  \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^n f(x_i^*) \Delta x.
  \]
- The arc length of a curve \( r : [t_0, t_1] \to \mathbb{R}^3 \) in space is
  \[
  s_{t_1,t_0} = \int_{t_0}^{t_1} |r'(t)| \, dt.
  \]
- The integral of a function \( f : \mathbb{R}^3 \to \mathbb{R} \) along a curve
  \( r : [t_0, t_1] \to \mathbb{R}^3 \) is
  \[
  \int_C f \, ds = \int_{t_0}^{t_1} f(r(t)) |r'(t)| \, dt.
  \]
- The circulation of a function \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) along a curve
  \( r : [t_0, t_1] \to \mathbb{R}^3 \) is
  \[
  \int_C F \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} F(r(t)) \cdot r'(t) \, dt.
  \]
- The flux of a function \( F : \{z = 0\} \cap \mathbb{R}^3 \to \{z = 0\} \cap \mathbb{R}^3 \) along a loop \( r : [t_0, t_1] \to \{z = 0\} \cap \mathbb{R}^3 \) is
  \[
  \Phi = \oint_C F \cdot \mathbf{n} \, ds.
  \]
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Review: Double integral of a scalar function

- The double integral of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on a region $R \subset \mathbb{R}^2$, which is the volume under the graph of $f$ and above the $z = 0$ plane, and is given by

$$
\iint_R f \, dA = \lim_{n \to \infty} \sum_{i=0}^n \sum_{j=0}^n f(x_i^*, y_j^*) \Delta x \Delta y.
$$

- The area of a flat surface $R \subset \mathbb{R}^2$ is the particular case $f = 1$, that is, $A(R) = \iint_R dA$.

We will show how to compute:

- The area of a **non-flat surface** in space. (Today.)
- The integral of a scalar function $f$ on a surface is space.
- The flux of a vector-valued function $\mathbf{F}$ on a surface in space.
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Explicit, implicit, parametric equations of surfaces

**Review:** Curves on $\mathbb{R}^2$ can be defined in:

- Explicit form, $y = f(x)$;
- Implicit form, $F(x, y) = 0$;
- Parametric form, $r(t) = (x(t), y(t))$.

The vector $r'(t) = (x'(t), y'(t))$ is tangent to the curve.

**Review:** Surfaces in $\mathbb{R}^3$ can be defined in:

- Explicit form, $z = f(x, y)$;
- Implicit form, $F(x, y, z) = 0$;
- Parametric form, $r(u, v) = (x(u, v), y(u, v), z(u, v))$.

Two vectors tangent to the surface are

$$
\partial_u r(u, v) = (\partial_u x(u, v), \partial_u y(u, v), \partial_u z(u, v)),
$$

$$
\partial_v r(u, v) = (\partial_v x(u, v), \partial_v y(u, v), \partial_v z(u, v)).
$$
Explicit, implicit, parametric equations of surfaces

Example
Find a parametric expression for the cone $z = \sqrt{x^2 + y^2}$, and two tangent vectors.

Solution: Use cylindrical coordinates: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $z = z$. Parameters of the surface: $u = r$, $v = \theta$. Then

$$x(r, \theta) = r \cos(\theta), \quad y(r, \theta) = r \sin(\theta), \quad z(r, \theta) = r.$$

Using vector notation, a parametric equation of the cone is

$$\mathbf{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), r \rangle.$$

Two tangent vectors to the cone are $\partial_r \mathbf{r}$ and $\partial_\theta \mathbf{r}$,

$$\partial_r \mathbf{r} = \langle \cos(\theta), \sin(\theta), 1 \rangle, \quad \partial_\theta \mathbf{r} = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle.$$  

Example
Find a parametric expression for the sphere $x^2 + y^2 + z^2 = R^2$, and two tangent vectors.

Solution: Use spherical coordinates:

$$x = \rho \cos(\theta) \sin(\phi), \quad y = \rho \sin(\theta) \sin(\phi), \quad z = \rho \cos(\phi).$$

Parameters of the surface: $u = \theta$, $v = \phi$.

$$x = R \cos(\theta) \sin(\phi), \quad y = R \sin(\theta) \sin(\phi), \quad z = R \cos(\phi).$$

Using vector notation, a parametric equation of the cone is

$$\mathbf{r}(\theta, \phi) = R \langle \cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi) \rangle.$$

Two tangent vectors to the paraboloid are $\partial_\theta \mathbf{r}$ and $\partial_\phi \mathbf{r}$,

$$\partial_\theta \mathbf{r} = R \langle -\sin(\theta) \sin(\phi), \cos(\theta) \sin(\phi), 0 \rangle,$$

$$\partial_\phi \mathbf{r} = R \langle \cos(\theta) \cos(\phi), \sin(\theta) \cos(\phi), -\sin(\phi) \rangle.$$  

◮
The area of a surface in parametric form

**Theorem**

*Given a smooth surface* $S$ *with parametric equation*

$$
\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \quad \text{for} \quad u \in [u_0, u_1] \quad \text{and} \quad v \in [v_0, v_1]
$$

*is given by*

$$
A(S) = \int_{u_0}^{u_1} \int_{v_0}^{v_1} |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.
$$

**Remark:** The function

$$
d\sigma = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.
$$

represents the area of a small region on the surface.

This is the generalization to surfaces of the arc-length formula for the length of a curve.
The area of a surface in parametric form

Example
Find an expression for the area of the surface in space given by the paraboloid \( z = x^2 + y^2 \) between the planes \( z = 0 \) and \( z = 4 \).

Solution: Use cylindrical coordinates. The surface in parametric form is

\[ r(r, \theta) = (r \cos(\theta), r \sin(\theta), r^2). \]

The tangent vectors to the surface \( \partial_r r, \partial_\theta r \) are

\[ \partial_r r = (\cos(\theta), \sin(\theta), 2r), \quad \partial_\theta r = (-r \sin(\theta), r \cos(\theta), 0). \]

\[ \partial_r r \times \partial_\theta r = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle. \]

The area of a surface in parametric form

Example
Find an expression for the area of the surface in space given by the paraboloid \( z = x^2 + y^2 \) between the planes \( z = 0 \) and \( z = 4 \).

Solution: Recall: \( \partial_r r \times \partial_\theta r = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle \).

\[ |\partial_r r \times \partial_\theta r| = \sqrt{4r^4 + r^2} = r \sqrt{1 + 4r^2}. \]

\[ A(S) = \int_0^{2\pi} \int_0^2 r \sqrt{1 + 4r^2} \, dr \, d\theta. \]

This integral will be done later on by substitution. The result is:

\[ A(S) = \frac{\pi}{6} [(17)^{3/2} - 1]. \]
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**The area of a surface in space in explicit form**

**Theorem**

*Given a smooth function \( f : \mathbb{R}^3 \to \mathbb{R} \), the area of a level surface \( S = \{ f(x, y, z) = 0 \} \), over a closed, bounded region \( R \) in the plane \( \{ z = 0 \} \), is given by*

\[
A(S) = \iint_{R} \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.
\]

**Remark:** Eq. (7), page 949, in the textbook is more general than the equation above, since the region \( R \) can be located on any plane, not only the plane \( \{ z = 0 \} \) considered here.

The vector \( \mathbf{p} \) in the textbook is the vector normal to \( R \). In our case \( \mathbf{p} = \mathbf{k} \).
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat region $R$ in $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.$$  

Example
Find the area of $S = \{z - 1 = 0\}$ over $R$ in $\{z = 0\}$.

Solution: This is simple: $f(x, y, z) = z - 1$, so $\nabla f = \mathbf{k}$, hence

$$\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = 1 \quad \Rightarrow \quad A(S) = \iint_R \, dx \, dy = A(R). \quad \triangle$$

Remark: The formula for $A(S)$ is reasonable: Every flat horizontal surface $S$ over a flat horizontal region $R$ satisfies $A(S) = A(R)$.

---

The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat region $R$ in $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.$$  

Example
Find the area of $S = \{y + z - 1 = 0\}$ over $R$ in $\{z = 0\}$.

Solution: The plane $S$ intersects the horizontal plane at a $\pi/4$ angle. So, $f(x, y, z) = y + z - 1$, and $\nabla f = \mathbf{j} + \mathbf{k}$, hence

$$\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \sqrt{2} \quad \Rightarrow \quad A(S) = \iint_R \sqrt{2} \, dx \, dy \Rightarrow A(S) = \sqrt{2} A(R). \quad \triangle$$

Remark: The formula for $A(S)$ is still reasonable: Every flat surface $S$ having an angle $\pi/4$ over a flat horizontal region $R$ satisfies $A(S) = \sqrt{2} A(R)$. 
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat horizontal region $R$ in $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.$$ 

Remark: The formula for $A(S)$ can be interpreted as follows:

The factor $\frac{|\nabla f|}{|\nabla f \cdot k|}$ is the angle correction function needed to obtain the $A(S)$ by correcting the $A(R)$ by the relative inclination of $S$ with respect to $R$.

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The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution:

The surface is given by $f = 0$ with

$$f(x, y, z) = x + 2y + 2z - 5.$$ 

The region $R$ is in the plane $z = 0$,

$$R = \left\{ (x, y, z) : z = 0, \quad y \in [-1, 1] \right\} \text{ with } x \in [y^2, (2 - y^2)].$$ 

Recall: $A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA$. Here $\nabla f = \langle 1, 2, 2 \rangle$. 

\[\nabla f = \langle 1, 2, 2 \rangle.\]
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \int\int_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} dA$. Here $\nabla f = (1, 2, 2)$.

Therefore: $|\nabla f| = \sqrt{1 + 4 + 4} = 3$, and $|\nabla f \cdot k| = 2$.

And the region $R = \{(x, y) : y \in [-1, 1], x \in [y^2, (2 - y^2)]\}$.

So we can write down the expression for $A(S)$ as follows,

$$A(S) = \int\int_{R} \frac{3}{2} dx dy = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx dy.$$

The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx dy$.

$$A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) dy$$

$$A(S) = 3 \int_{-1}^{1} (1 - y^2) dy = 3\left(y - \frac{y^3}{3}\right)\bigg|_{-1}^{1} = 3\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$$A(S) = 3\left(2 - \frac{2}{3}\right) = 3 \cdot \frac{4}{3} \Rightarrow A(S) = 4.$$
The area of a surface in space in explicit form

Example
Find the area of the surface in space given by the paraboloid
\[ z = x^2 + y^2 \] between the planes \( z = 0 \) and \( z = 4 \).

Solution: The surface is the level surface of the function
\[ f(x, y, z) = x^2 + y^2 - z. \] The region \( R \) is the disk \( z = x^2 + y^2 \leq 4 \).

\[ A(S) = \int \int_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dx \, dy, \quad \nabla f = \langle 2x, 2y, -1 \rangle, \quad \nabla f \cdot k = -1, \]

\[ A(S) = \int \int_R \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy. \]

Since \( R \) is a disk radius 2, it is convenient to use polar coordinates in \( \mathbb{R}^2 \). We obtain

\[ A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta. \]

We conclude:

\[ A(S) = \frac{\pi}{6} \left[ (17)^{3/2} - 1 \right]. \]

\[ \triangleleft \]
**The area of a surface in space in explicit form**

**Remark**: The formula for the area of a surface in space can be generalized as follows.

**Theorem**

The area of a surface $S$ given by $f(x, y, z) = 0$ over a closed and bounded plane region $R$ in space is given by

$$A(S) = \int \int_{R} \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} \, dA,$$

where $\mathbf{p}$ is a unit vector normal to the region $R$ and $\nabla f \cdot \mathbf{p} \neq 0$.

**Proof in a simple case**: Assume that the surface is given in explicit form:

$$S = \{(x, y, z) : z = g(x, y)\},$$

On the one hand, a simple parametric form is to use $u = x$, $v = y$ and $z(u, v) = g(u, v)$. Hence

$$\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle \Rightarrow \begin{cases} \partial_x \mathbf{r} = \langle 1, 0, \partial_x g \rangle \\ \partial_y \mathbf{r} = \langle 0, 1, \partial_y g \rangle, \end{cases}$$

$$\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1 \rangle$$

On the other hand, an implicit form for the surface is

$$f(x, y, z) = g(x, y) - z$$

Therefore, $\partial_x f = \partial_x g$, $\partial_y f = \partial_y g$, $\partial_z f = -1$. 
The area of a surface in space in explicit form

Proof in a simple case: Recall: \( \partial_x r \times \partial_y r = (-\partial_x g, -\partial_y g, 1) \) and

\[
\partial_x f = \partial_x g, \quad \partial_y f = \partial_y g, \quad \partial_z f = -1.
\]

One can show (with chain rule) that \( \partial_x r \times \partial_y r \) is given by

\[
\partial_x r \times \partial_y r = \frac{\nabla f}{\nabla f \cdot k}.
\]

That is, \( \partial_x r \times \partial_y r = \frac{\nabla f}{\nabla f \cdot k} \). We then obtain

\[
A(S) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} |\partial_x r \times \partial_y r| \, dy \, dx = \int_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.
\]

The area of a surface in space in explicit form

Proof: Introduce a partition in \( R \subset \mathbb{R}^2 \), and consider an arbitrary rectangle \( \Delta R \) in that partition. We compute the area \( \Delta P \).

It is simple to see that

\[
\Delta P = |u \times v|,
\]

and

\[
u = \langle \Delta x, 0, (z_i - \hat{z}_i) \rangle, \quad v = \langle 0, \Delta y, (z_i - \bar{z}_i) \rangle.
\]

Therefore,

\[
u \times v = \begin{vmatrix}
i & j & k \\
\Delta x & 0 & (z_i - \hat{z}_i) \\
0 & \Delta y & (z_i - \bar{z}_i)
\end{vmatrix} = -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y).\]
The area of a surface in space in explicit form

Proof: Recall: \( \mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - z_i), \Delta x\Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \( (x_i, y_i, z_i) \) implies
\[
f(x, y, z) \simeq f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).
\]
Since \( f(x_i, y_i, z_i) = 0 \), \( f(x_i + \Delta x, y_i, \hat{z}_i) = 0 \), \( f(x_i, y_i + \Delta y, \bar{z}_i) = 0 \),
\[
0 = (\partial_x f)_i \Delta x + (\partial_z f)_i (z_i - \hat{z}_i) \Rightarrow (z_i - \hat{z}_i) = -\frac{(\partial_x f)_i}{(\partial_z f)_i} \Delta x,
\]
\[
0 = (\partial_y f)_i \Delta y + (\partial_z f)_i (z_i - \bar{z}_i) \Rightarrow (z_i - \bar{z}_i) = -\frac{(\partial_y f)_i}{(\partial_z f)_i} \Delta y.
\]
\[
\mathbf{u} \times \mathbf{v} = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \frac{\Delta x\Delta y}{(\partial_z f)_i} \Rightarrow \mathbf{u} \times \mathbf{v} = \frac{(\nabla f)_i}{(\nabla f \cdot \mathbf{k})_i} \Delta x\Delta y.
\]
\[
\Delta P = \frac{|(\nabla f)_i|}{|\nabla f \cdot \mathbf{k}|_i} \Delta x\Delta y \Rightarrow A(S) = \iint_{R} \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dA. \quad \square
\]