Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- Divergence and curl of a function on a plane.
- Area computed with a line integral.

Review: Green's Theorem on a plane

Theorem

Given a field $\mathbf{F} = \langle F_x, F_y \rangle$ and a loop C enclosing a region $R \in \mathbb{R}^2$ described by the function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \in [t_0, t_1]$, with unit tangent vector $\mathbf{u}$ and exterior normal vector $\mathbf{n}$, then holds:

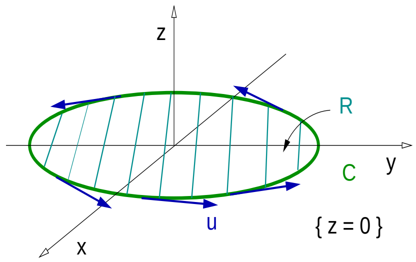
- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds$ *satisfies:*

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *satisfies:*

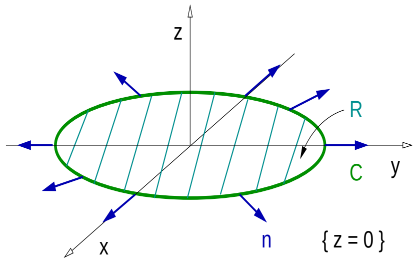
$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$

Review: Green's Theorem on a plane



Circulation-tangential form:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy.$$



Flux-normal form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy.$$

Theorem

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

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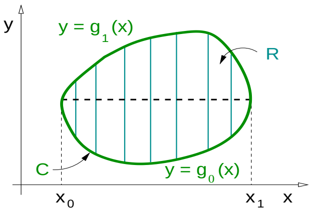
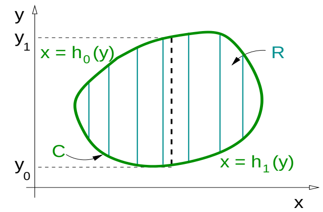
Sketch of the proof of Green's Theorem

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \left[F_x(t)\, x'(t) + F_y(t)\, y'(t)\right] dt = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy.$$

We only consider a simple domain like the one in the pictures.





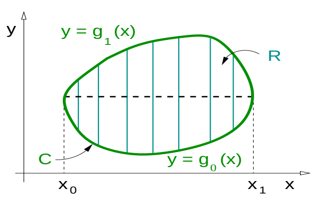
Using the picture on the left we show that

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

and using the picture on the right we show that

$$\int_C F_y(t)\, y'(t)\, dt = \iint_R (\partial_x F_y)\, dx\, dy.$$

Sketch of the proof of Green's Theorem



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

The path C can be described by the curves $\mathbf{r}_0$ and $\mathbf{r}_1$ given by

$$\mathbf{r}_0(t) = \langle t, g_0(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Therefore,

$$\mathbf{r}_0'(t) = \langle 1, g_0'(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Recall: $F_x(t) = F_x(t, g_0(t))$ on $\mathbf{r}_0$,
and $F_x(t) = F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))$ on $\mathbf{r}_1$.

Sketch of the proof of Green's Theorem

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

$$- \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt$$

Substitution in the second term: $\tau = x_1 + x_0 - t$, so $d\tau = -dt$.

$$- \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt =$$

$$- \int_{x_1}^{x_0} F_x(\tau, g_1(\tau))\,(-d\tau) = - \int_{x_0}^{x_1} F_x(\tau, g_1(\tau))\,d\tau.$$

Therefore, $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \big[F_x(t, g_0(t)) - F_x(t, g_1(t))\big]\,dt.$

We obtain: $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t, y)\big]\,dy\,dt.$

Sketch of the proof of Green's Theorem

Recall: $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t, y)\big]\,dy\,dt.$

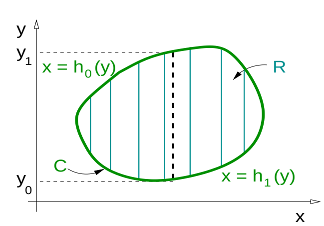
This result is precisely what we wanted to prove:

$$\int_C F_x(t)x'(t)\,dt = \iint_R (-\partial_y F_x)\,dy\,dx.$$

We just mention that the result

$$\int_C F_y(t)\,y'(t)\,dt = \iint_R (\partial_x F_y)\,dx\,dy.$$

is proven in a similar way using the parametrization of the C given in the picture.



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Divergence and curl of a function on a plane

Definition
The *curl* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\big(\mathrm{curl}\,\mathbf{F}\big)_z = \partial_x F_y - \partial_y F_x.$$

The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\mathrm{div}\,\mathbf{F} = \partial_x F_x + \partial_y F_y.$$

Remark: Both forms of Green's Theorem can be written as:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \big(\mathrm{curl}\,\mathbf{F}\big)_z\, dx\, dy.$$

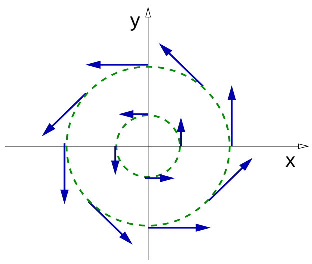
$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \mathrm{div}\,\mathbf{F}\, dx\, dy.$$

Divergence and curl of a function on a plane

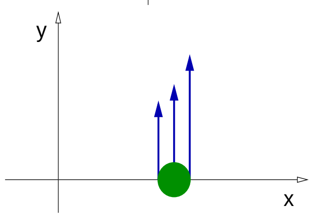
Remark: What type of information about **F** is given in $(\operatorname{curl} \mathbf{F})_z$?

Example: Suppose **F** is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad (\operatorname{curl} \mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2.$$



If we place a small ball at $(0,0)$, the ball will spin around the z-axis with speed proportional to $(\operatorname{curl} \mathbf{F})_z$.



If we place a small ball at everywhere in the plane, the ball will spin around the z-axis with speed proportional to $(\operatorname{curl} \mathbf{F})_z$.

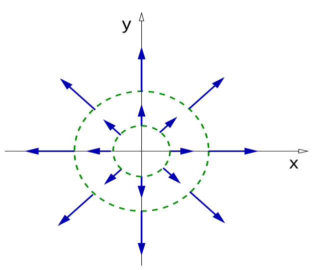
Remark: The curl of a field measures its rotation.

Divergence and curl of a function on a plane

Remark: What type of information about **F** is given in $\operatorname{div} \mathbf{F}$?

Example: Suppose **F** is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$



The field **F** represents the gas as is heated with a heat source at $(0,0)$. The heated gas expands in all directions, radially out form $(0,0)$. The $\operatorname{div} \mathbf{F}$ measures that expansion.

Remark: The divergence of a field measures its expansion.

Remarks:

- Notice that for $\mathbf{F} = \langle x, y \rangle$ we have $(\operatorname{curl} \mathbf{F})_z = 0$.
- Notice that for $\mathbf{F} = \langle -y, x \rangle$ we have $\operatorname{div} \mathbf{F} = 0$.

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Area computed with a line integral

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy = \oint_C (F_x\, dy - F_y\, dx)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C x\, dy.$$

$$\mathbf{F} = \langle 0, y\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C -y\, dx.$$

$$\mathbf{F} = \frac{1}{2}\langle x, y\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \frac{1}{2}\oint_C (x\, dy - y\, dx).$$

Area computed with a line integral

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt = \int_0^{2\pi} a\cos(t)\,b\cos(t)\,dt.$$

$$A(R) = ab\int_0^{2\pi} \cos^2(t)\,dt = ab\int_0^{2\pi} \frac{1}{2}\big[1+\cos(2t)\big]\,dt.$$

Since $\displaystyle\int_0^{2\pi} \cos(2t)\,dt = 0$, we obtain $A(R) = \dfrac{ab}{2}\,2\pi$, that is,

$$A(R) = \pi ab.$$

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