Review: The line integral of a vector field along a curve

**Definition**

The *line integral* of a vector-valued function \( \mathbf{F} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), along the curve \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3 \), with arc length function \( s \), is given by

\[
\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt,
\]

where \( \mathbf{u} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \), and \( s_0 = s(t_0) \), \( s_1 = s(t_1) \).

**Example**

**Remark:** Since \( \mathbf{F} = \langle F_x, F_y \rangle \) and \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \), in components,

\[
\int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt = \int_{t_0}^{t_1} \left[ F_x(t)x'(t) + F_y(t)y'(t) \right] \, dt.
\]
**Review: The line integral of a vector field along a curve**

**Example**
Evaluate the line integral of \( \mathbf{F} = \langle -y, x \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

**Solution:** Evaluate \( \mathbf{F} \) along the curve: 
\[
\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle.
\]
Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle \).
Then evaluate the line integral in components,
\[
\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} [F_x(t)x'(t) + F_y(t)y'(t)] \, dt,
\]
\[
\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{u} \, ds = \int_{0}^{2\pi} [(-\sin(t))(-\sin(t)) + \cos(t)\cos(t)] \, dt,
\]
\[
\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{u} \, ds = \int_{0}^{2\pi} [\sin^2(t) + \cos^2(t)] \, dt \Rightarrow \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi.
\]

**Review: The flux across a plane loop**

**Definition**
The **flux** of a vector field \( \mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3 \) is given by
\[
\mathbb{F} = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n} \, ds,
\]
where \( \mathbf{n} \) is the unit outer normal vector to the curve inside the plane \( \{z = 0\} \).

**Example**

\[
\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} [F_x(t)y'(t) - F_y(t)x'(t)] \, dt.
\]

**Remark:** Since \( \mathbf{F} = \langle F_x, F_y, 0 \rangle \),
\[
\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle, \quad ds = |\mathbf{r}'(t)| \, dt, \quad \text{and} \quad \mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle, \quad \text{in components},
\]
\[
\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} [F_x(t)y'(t) - F_y(t)x'(t)] \, dt.
\]
Example
Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the normal vector \( \mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle \), that is, \( \mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle \). Evaluate the flux integral in components,
\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(y'(t)) - F_y(x'(t)) \right] \, dt,
\]
\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} \left[ -\sin(t) \cos(t) - \cos(t)(-\sin(t)) \right] \, dt,
\]
\[
\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_0^{2\pi} 0 \, dt \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot \mathbf{u} \, ds = 0. \quad \triangleleft
\]

Green’s Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- **Green’s Theorem on a plane.**
  - Circulation-tangential form.
  - Flux-normal form.
- Tangential and normal forms equivalence.
Green’s Theorem on a plane

Theorem (Circulation-tangential form)

The counterclockwise line integral \( \oint_C \mathbf{F} \cdot \mathbf{u} \, ds \) of the field \( \mathbf{F} = \langle F_x, F_y \rangle \) along a loop \( C \) enclosing a region \( R \subset \mathbb{R}^2 \) and given by the function \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) and with unit tangent vector \( \mathbf{u} \), satisfies that

\[
\int_{t_0}^{t_1} [F_x(t)x'(t) + F_y(t)y'(t)] \, dt = \iint_R (\partial_x F_y - \partial_y F_x) \, dx \, dy.
\]

Equivalently,

\[
\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \iint_R (\partial_x F_y - \partial_y F_x) \, dx \, dy.
\]

Example

Verify Green’s Theorem tangential form for the field \( \mathbf{F} = \langle -y, x \rangle \) and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Recall: We found that \( \oint_C \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi \).

Now we compute the double integral \( I = \iint_R (\partial_x F_y - \partial_y F_x) \, dx \, dy \) and we verify that we get the same result, \( 2\pi \).

\[
I = \iint_R [1 - (-1)] \, dx \, dy = 2 \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = 2 \int_0^{2\pi} \left. \frac{r^2}{2} \right|_0^1 = 2 \pi.
\]

We verified that \( \oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \iint_R (\partial_x F_y - \partial_y F_x) \, dx \, dy = 2\pi \). \( \checkmark \)
Green’s Theorem on a plane. (Sect. 16.4)

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**Theorem (Flux-normal form)**

The counterclockwise flux integral \( \oint_C F \cdot n \, ds \) of the field \( F = \langle F_x, F_y \rangle \) along a loop \( C \) enclosing a region \( R \subset \mathbb{R}^2 \) and given by the function \( r(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) and with unit normal vector \( n \), satisfies that

\[
\int_{t_0}^{t_1} \left[ F_x(t) y'(t) - F_y(t) x'(t) \right] \, dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy.
\]

Equivalently,

\[
\oint_C F \cdot n \, ds = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy.
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle -y, x \rangle \) and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Recall: We found that \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 0 \).

Now we compute the double integral \( I = \iint_R (\partial_x F_x + \partial_y F_y) \, dx \, dy \) and we verify that we get the same result, 0.

\[
I = \iint_R \left[ \partial_x (-y) + \partial_y (x) \right] \, dx \, dy = \iint_R 0 \, dx \, dy = 0.
\]

We verified that \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (\partial_x F_x + \partial_y F_y) \, dx \, dy = 0 \). \quad \triangledown

Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

Solution: We start with the line integral

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]

It is simple to see that \( \mathbf{F}(t) = \langle 2a \cos(t), -3a \sin(t) \rangle \), and also that \( \mathbf{r}'(t) = \langle -a \sin(t), a \cos(t) \rangle \).

Therefore, \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ 2a^2 \cos^2(t) - 3a^2 \sin^2(t) \right] \, dt \),

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ 2a^2 \frac{1}{2} (1 + \cos(2t)) - 3a^2 \frac{1}{2} (1 - \cos(2t)) \right] \, dt.
\]

Since \( \int_{0}^{2\pi} \cos(2t) \, dt = 0 \), we conclude \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = -\pi a^2 \).
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi], \ a > 0 \).

Solution: Recall: \( \oint_C \mathbf{F} \cdot n \, ds = -\pi a^2 \).

Now we compute the double integral \( I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy \).

\[
I = \iint_R \left[ \partial_x (2x) + \partial_y (-3y) \right] \, dx \, dy = \iint_R (2 - 3) \, dx \, dy.
\]

\[
I = -\int_0^{2\pi} \int_0^a r \, dr \, d\theta = -2\pi \left( \frac{r^2}{2} \right) \bigg|_0^a = -\pi a^2.
\]

Hence, \( \oint_C \mathbf{F} \cdot n \, ds = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy = -\pi a^2 \).

\[
\triangleq
\]

Green’s Theorem on a plane. (Sect. 16.4)

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Tangential and normal forms equivalence

Theorem

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green’s Theorem in tangential form for \( \hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle \) says

\[
\int_{t_0}^{t_1} [\hat{F}_x(t)x'(t) + \hat{F}_y(t)y'(t)] \, dt = \iint_R (\partial_x \hat{F}_y - \partial_y \hat{F}_x) \, dx \, dy.
\]

If \( \hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle \) and \( \mathbf{F} = \langle F_x, F_y \rangle \) are related by \( \hat{F}_x = -F_y \) and \( \hat{F}_y = F_x \), then the equation above for \( \hat{\mathbf{F}} \) written in terms of \( \mathbf{F} \) is

\[
\int_{t_0}^{t_1} [-F_y(t)x'(t) + F_x(t)y'(t)] \, dt = \iint_R \left( \partial_x F_x - \partial_y (-F_y) \right) \, dx \, dy,
\]

so,

\[
\int_{t_0}^{t_1} [F_x(t)y'(t) - F_y(t)x'(t)] \, dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy,
\]

which is Green’s Theorem in normal form for \( \mathbf{F} \). The converse implication is proved in the same way. □

Using Green’s Theorem

Example

Use Green’s Theorem to find the counterclockwise circulation of the field \( \mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle \) along the curve \( C \) that is the triangle bounded by \( y = 0 \), \( x = 3 \) and \( y = x \).

Solution: Recall: \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\partial_x F_y - \partial_y F_x) \, dx \, dy \).

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y) \, dx \, dy = \int_0^3 \int_0^x (2x - 2y) \, dy \, dx,
\]

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[ 2x \left( y \bigg|_0^x \right) - \left( y^2 \bigg|_0^x \right) \right] \, dx = \int_0^3 \left( 2x^2 - x^2 \right) \, dx,
\]

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 x^2 \, dx = \frac{x^3}{3} \bigg|_0^3 = \oint_C \mathbf{F} \cdot d\mathbf{r} = 9. \quad \triangleq
\]