Green's Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- Green's Theorem on a plane.
  - Circulation-tangential form.
  - Flux-normal form.
- Tangential and normal forms equivalence.

Review: The line integral of a vector field along a curve

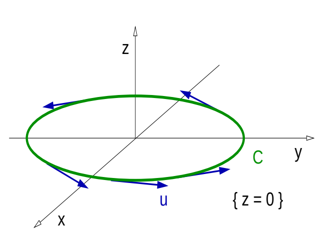
Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, with arc length function s, is given by

$$\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{u} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$, and $s_0 = s(t_0)$, $s_1 = s(t_1)$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y \rangle$ and $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, in components,

$$\int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_{t_0}^{t_1} \big[F_x(t) x'(t) + F_y(t) y'(t) \big]\, dt.$$

Review: The line integral of a vector field along a curve

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.
Then evaluate the line integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \big[F_x(t)x'(t) + F_y(t)y'(t)\big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[(-\sin(t))(-\sin(t)) + \cos(t)\cos(t)\big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[\sin^2(t) + \cos^2(t)\big]\, dt \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi.$$

$\triangleleft$

Review: The flux across a plane loop

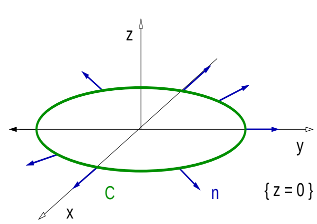
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y, 0 \rangle$, $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$, $ds = |\mathbf{r}'(t)|\, dt$, and $\mathbf{n} = \dfrac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle$, in components,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t)y'(t) - F_y(t)x'(t)\big]\, dt.$$

Review: The flux across a plane loop

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0\rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0\rangle$. Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0\rangle$. Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0\rangle$, that is, $\mathbf{n}(t) = \langle \cos(t), \sin(t), 0\rangle$. Evaluate the flux integral in components,

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \big[F_x(t)y'(t) - F_y(t)x'(t)\big]\,dt,$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-\sin(t)\cos(t) - \cos(t)(-\sin(t))\big]\,dt,$$

$$\oint_C \mathbf{F}\cdot\mathbf{u}\,ds = \int_0^{2\pi} 0\,dt \quad\Rightarrow\quad \oint_C \mathbf{F}\cdot\mathbf{u}\,ds = 0. \qquad \lhd$$

Green's Theorem on a plane. (Sect. 16.4)

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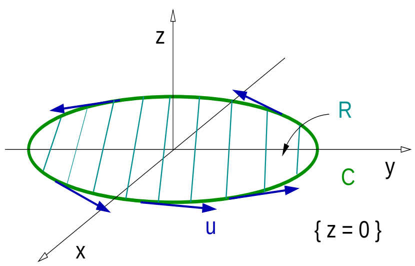
Green's Theorem on a plane

Theorem (Circulation-tangential form)

The counterclockwise line integral $\oint_C \mathbf{F}\cdot\mathbf{u}\,ds$ *of the field* $\mathbf{F} = \langle F_x, F_y\rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ *for* $t \in [t_0, t_1]$ *and with unit tangent vector* $\mathbf{u}$*, satisfies that*

$$\int_{t_0}^{t_1} \big[F_x(t)\,x'(t) + F_y(t)\,y'(t)\big]\,dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$$



Equivalently,

$$\oint_C \mathbf{F}\cdot\mathbf{u}\,ds = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$$

Green's Theorem on a plane

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x\rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F}\cdot\mathbf{u}\,ds = 2\pi$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\,dx\,dy = 2\iint_R dx\,dy = 2\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$I = 2(2\pi)\Big(\frac{r^2}{2}\Big|_0^1\Big) \quad\Rightarrow\quad I = 2\pi.$$

We verified that $\oint_C \mathbf{F}\cdot\mathbf{u}\,ds = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy = 2\pi.$ $\quad\triangleleft$

Green's Theorem on a plane. (Sect. 16.4)

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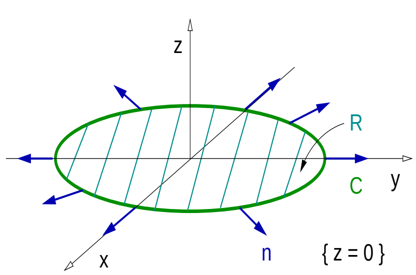
Green's Theorem on a plane

Theorem (Flux-normal form)

The counterclockwise flux integral $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit normal vector* $\mathbf{n}$*, satisfies that*

$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$



Equivalently,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$

Green's Theorem on a plane

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$ and we verify that we get the same result, 0.

$$I = \iint_R \big[\partial_x(-y) + \partial_y(x)\big]\, dx\, dy = \iint_R 0\, dx\, dy = 0.$$

We verified that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy = 0.$ $\triangleleft$

Green's Theorem on a plane

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t)\big]\, dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t) \rangle$,
and also that $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$.

Therefore, $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[2a^2\cos^2(t) - 3a^2\sin^2(t)\big]\, dt,$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \Big[2a^2\frac{1}{2}\big(1 + \cos(2t)\big) - 3a^2\frac{1}{2}\big(1 - \cos(2t)\big)\Big]\, dt.$$

Since $\int_0^{2\pi} \cos(2t)\, dt = 0$, we conclude $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Green's Theorem on a plane

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2.$

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy = \iint_R (2-3)\, dx\, dy.$$

$$I = -\iint_R dx\, dy = -\int_0^{2\pi}\int_0^a r\, dr\, d\theta = -2\pi\Big(\frac{r^2}{2}\Big|_0^a\Big) = -\pi a^2.$$

Hence, $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy = -\pi a^2.$ $\triangleleft$

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Tangential and normal forms equivalence

Theorem

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle$ says

$$\int_{t_0}^{t_1} \left[\hat{F}_x(t)\, x'(t) + \hat{F}_y(t)\, y'(t)\right] dt = \iint_R \left(\partial_x \hat{F}_y - \partial_y \hat{F}_x\right) dx\, dy.$$

If $\hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle$ and $\mathbf{F} = \langle F_x, F_y \rangle$ are related by $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$, then the equation above for $\hat{\mathbf{F}}$ written in terms of $\mathbf{F}$ is

$$\int_{t_0}^{t_1} \left[-F_y(t)\, x'(t) + F_x(t)\, y'(t)\right] dt = \iint_R \left(\partial_x F_x - \partial_y(-F_y)\right) dx\, dy,$$

so, $$\int_{t_0}^{t_1} \left[F_x(t)\, y'(t) - F_y(t)\, x'(t)\right] dt = \iint_R \left(\partial_x F_x + \partial_y F_y\right) dx\, dy,$$

which is Green's Theorem in normal form for $\mathbf{F}$. The converse implication is proved in the same way. □

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\partial_x F_y - \partial_y F_x\right) dx\, dy.$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[2x\left(y\Big|_0^x\right) - \left(y^2\Big|_0^x\right)\right] dx = \int_0^3 \left(2x^2 - x^2\right) dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 9. \qquad \triangleleft$$