# Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Gradient fields.
- Conservative fields.
- Equivalence of Gradient and Conservative fields.
- The line integral conservative fields.
- Finding the potential of a gradient field.
- Comments on exact differential forms.

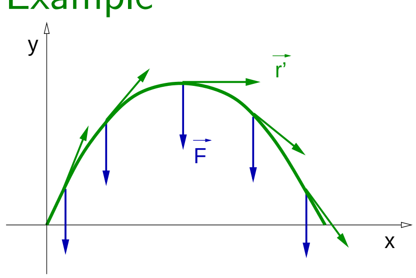
# The line integral of a vector field along a curve

Recall: The *line integral* of $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s)\, ds = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points $t_0$, $t_1$.

Example




Remark: It is common the notation

$$\hat{\mathbf{r}}' = \mathbf{T},$$

since $\mathbf{T}$ is tangent to the curve and unit, since $s$ is the curve arc-length parameter.

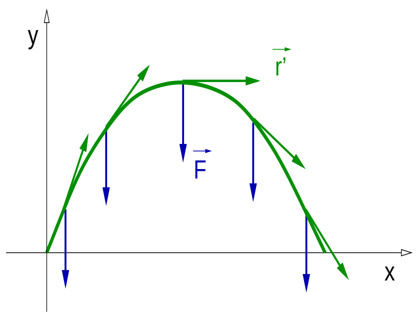
## Work done by a force on a particle

### Definition
In the case that **F** is a force on a particle with position function **r** then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

### Example




A projectile of mass $m$ moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leqslant 0$ in the first half of the trajectory, and $W \geqslant 0$ on the second half.

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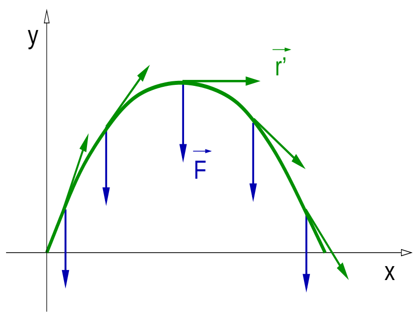
## Gradient fields

### Definition

A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called a *gradient field* iff there exists a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, called *potential function*, such that

$$\mathbf{F} = \nabla f.$$

### Example




A projectile of mass $m$ moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $\mathbf{F} = \nabla f$, with $f = -mgy$.

## Gradient fields

### Example

Show that the vector field $\mathbf{F} = \dfrac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is a gradient field and find the potential function.

Solution: The field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is a gradient field iff there exists a potential function $f$ such that $\mathbf{F} = \nabla f$, that is,

$$F_1 = \partial_{x_1} f, \qquad F_2 = \partial_{x_2} f, \qquad F_3 = \partial_{x_3} f.$$

Since

$$\frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = -\partial_{x_i} \Big[ \big( x_1^2 + x_2^2 + x_3^2 \big)^{-1/2} \Big], \quad i = 1, 2, 3,$$

then we conclude that $\mathbf{F} = \nabla f$, with $f = -\dfrac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$. $\lhd$

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## The line integral of conservative fields

### Definition

A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called a *conservative field* iff the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depend only on the initial and end points of the path.

Remark: For conservative fields is useful the following notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $\mathbf{r}_0$, $\mathbf{r}_1$, then denote the line integral of a conservative field $\mathbf{F}$ along $C$ as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$

Remarks:

- This notation emphasizes the end points, not the path.
- This notation is useful only for conservative fields.
- A field $\mathbf{F}$ is conservative iff $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.

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# Equivalence of Gradient and Conservative fields

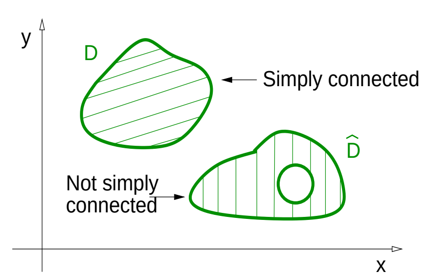
## Theorem (Equivalence of gradient and conservative fields)

- *A smooth vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, defined on a simply connected domain $D \subset \mathbb{R}^n$, is a gradient field iff it is a conservative field.*
- *Furthermore, if $\mathbf{F} = \nabla f$ and the curve $C \subset D$ starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$, then holds*

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).$$

Remarks:

- This is a Fundamental Theorem of Calculus for vector fields.
- A set is simply connected iff it consists of one piece and it contains no holes.



## Equivalence of Gradient and Conservative fields

Recall: A field $\mathbf{F}$ on a simply connected domain is a gradient field iff it is a conservative field. Furthermore,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).$$

Proof: Only ($\Rightarrow$).

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$. Therefore,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \frac{d}{dt}\big[f(\mathbf{r}(t)\big]\, dt = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)).$$

We conclude that $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).$ $\square$

(The statement ($\Leftarrow$) is more complicated to prove.)

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## The line integral of conservative fields

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: $I$ is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z\rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is a gradient field, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$. Therefore,

$$I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0) = |\mathbf{r}_1|^2 - |\mathbf{r}_0|^2 = (1 + 4 + 9).$$

We conclude that $I = 14$. $\triangleleft$

## The line integral of conservative fields (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path $C$ given by $\mathbf{r}(t) = \langle 1,2,3\rangle\, t$.
Then $\mathbf{r}(0) = \langle 0,0,0\rangle$, and $\mathbf{r}(1) = \langle 1,2,3\rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z\rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t\rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\,dt = \int_0^1 \langle 2t, 4t, 6t\rangle \cdot \langle 1,2,3\rangle\,dt$$

$$I = \int_0^1 (2t + 8t + 18t)\,dt = \int_0^1 28t\,dt = 28\left(\frac{t^2}{2}\Big|_0^1\right).$$

We conclude that $I = 14$. $\triangleleft$

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## Finding the potential of a gradient field

Theorem (Characterization of gradient fields)

*A smooth field* $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a gradient field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is a gradient field, there exists a scalar field $f$ such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j F_i.$$

$\square$

(The statement ($\Leftarrow$) is more complicated to prove.)

## Finding the potential of a gradient field

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is a gradient field.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$

$$\partial_3 F_1 = 0, \qquad \partial_1 F_3 = 0.$$

$\triangleleft$

## Finding the potential of a gradient field

Example

Find the potential of the gradient field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function $f$ solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).$$

$$\partial_z f = -2zy + \partial_z h(z) = -2yz \Rightarrow \partial_z h(z) = 0 \Rightarrow f = (x^2 - z^2)y + c_0.$$

$\triangleleft$

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## Comments on exact differential forms

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$

Remark: A differential form is a quantity that can be integrated along a path.

Definition

A differential form $\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz$ is called *exact* iff there exists a scalar function $f$ such that

$$F_x dx + F_y dy + F_z dz = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz.$$

Remarks:

- A differential form $\mathbf{F} \cdot d\mathbf{r}$ is exact iff $\mathbf{F} = \nabla f$.
- In this context an exact differential form is nothing else than another name for a gradient field.

## Comments on exact differential forms

**Example**

Show that the differential form given below is exact, where
$\mathbf{F} \cdot d\mathbf{r} = 2xy\, dx + (x^2 - z^2)\, dy - 2yz\, dz$ .

Solution: We need to do the same calculation we did above:
Writing $\mathbf{F} \cdot d\mathbf{r} = F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3$, show that

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. We showed that this is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$
$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$
$$\partial_3 F_1 = 0, \qquad \partial_1 F_3 = 0.$$

So, there exists $f$ such that $\mathbf{F} \cdot d\mathbf{r} = \nabla f \cdot d\mathbf{r}$. $\triangleleft$