The line integral of a vector field along a curve

Recall: The line integral of $F : \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along $r : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$
\int_C F \cdot dr = \int_{s_0}^{s_1} F(\hat{r}(s)) \cdot \hat{r}'(s) \, ds = \int_{t_0}^{t_1} F(r(t)) \cdot r'(t) \, dt,
$$

where $\hat{r}(s)$ is the arc length parametrization of the function $r$, and $s(t_0) = s_0, s(t_1) = s_1$ are the arc lengths at the points $t_0, t_1$.

**Example**

**Remark:** It is common the notation $\hat{r}' = T$, since $T$ is tangent to the curve and unit, since $s$ is the curve arc-length parameter.
Work done by a force on a particle

**Definition**
In the case that $F$ is a force on a particle with position function $r$ then the line integral

$$W = \int_C F \cdot dr,$$

is called the *work* done by the force on the particle.

**Example**

A projectile of mass $m$ moving on the surface of Earth.

- The movement takes place on a plane, and $F = \langle 0, -mg \rangle$.
- $W \leq 0$ in the first half of the trajectory, and $W \geq 0$ on the second half.

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- **Gradient fields.**
  - Conservative fields.
  - Equivalence of Gradient and Conservative fields.
  - The line integral conservative fields.
  - Finding the potential of a gradient field.
  - Comments on exact differential forms.
Gradient fields

Definition
A vector field \( \mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), is called a gradient field iff there exists a scalar function \( f : D \subset \mathbb{R}^n \to \mathbb{R} \), called potential function, such that

\[
\mathbf{F} = \nabla f.
\]

Example
A projectile of mass \( m \) moving on the surface of Earth.

\[\begin{align*}
\text{▶} & \quad \text{The movement takes place on a plane, and } \mathbf{F} = \langle 0, -mg \rangle. \\
\text{▶} & \quad \mathbf{F} = \nabla f, \text{ with } f = -mgy.
\end{align*}\]

Gradient fields

Example
Show that the vector field \( \mathbf{F} = \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle \) is a gradient field and find the potential function.

Solution: The field \( \mathbf{F} = \langle F_1, F_2, F_3 \rangle \) is a gradient field iff there exists a potential function \( f \) such that \( \mathbf{F} = \nabla f \), that is,

\[
F_1 = \partial_{x_1} f, \quad F_2 = \partial_{x_2} f, \quad F_3 = \partial_{x_3} f.
\]

Since

\[
\frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = -\partial_{x_i} \left[ (x_1^2 + x_2^2 + x_3^2)^{-1/2} \right], \quad i = 1, 2, 3,
\]

then we conclude that \( \mathbf{F} = \nabla f \), with \( f = -\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \).

\( \triangleright \)
The line integral of conservative fields

Definition
A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called a \textit{conservative field} iff the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depend only on the initial and end points of the path.

Remark: For conservative fields is useful the following notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $r_0, r_1$, then denote the line integral of a conservative field $\mathbf{F}$ along $C$ as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{r_0}^{r_1} \mathbf{F} \cdot d\mathbf{r}.$$  

Remarks:
- This notation emphasizes the end points, not the path.
- This notation is useful only for conservative fields.
- A field $\mathbf{F}$ is conservative iff $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.
Conservative fields and potential functions. (Sect. 16.3)

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**Equivalence of Gradient and Conservative fields**

**Theorem (Equivalence of gradient and conservative fields)**

- A smooth vector field \( \mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), defined on a simply connected domain \( D \subset \mathbb{R}^n \), is a gradient field iff it is a conservative field.

- Furthermore, if \( \mathbf{F} = \nabla f \) and the curve \( C \subset D \) starts at \( \mathbf{r}_0 \) and ends at \( \mathbf{r}_1 \), then holds

\[
\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).
\]

**Remarks:**

- This is a Fundamental Theorem of Calculus for vector fields.
- A set is simply connected iff it consists of one piece and it contains no holes.
Equivalence of Gradient and Conservative fields

Recall: A field \( \mathbf{F} \) on a simply connected domain is a gradient field iff it is a conservative field. Furthermore,

\[
\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).
\]

Proof: Only \( \Rightarrow \).

\[
\int_{c} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f) \cdot \mathbf{r}'(t) \, dt,
\]

where \( \mathbf{r}(t_0) = \mathbf{r}_0 \) and \( \mathbf{r}(t_1) = \mathbf{r}_1 \). Therefore,

\[
\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \frac{d}{dt}[f(\mathbf{r}(t))] \, dt = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)).
\]

We conclude that \( \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0) \). \( \square \)

(The statement \( \Leftarrow \) is more complicated to prove.)

Conservative fields and potential functions. (Sect. 16.3)

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The line integral of conservative fields

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} (2x \, dx + 2y \, dy + 2z \, dz) \) along a straight line.

Solution: \( I \) is a line integral for a field in \( \mathbb{R}^3 \), since

\[
I = \int_{(0,0,0)}^{(1,2,3)} (2x, 2y, 2z) \cdot (dx, dy, dz).
\]

Introduce \( F = (2x, 2y, 2z) \), \( r_0 = (0, 0, 0) \) and \( r_1 = (1, 2, 3) \), then

\[
I = \int_{r_0}^{r_1} F \cdot dr. \quad \text{The field } F \text{ is a gradient field, since } F = \nabla f \text{ with potential } f(x, y, z) = x^2 + y^2 + z^2. \quad \text{That is } f(r) = |r|^2. \quad \text{Therefore,}
\]

\[
I = \int_{r_0}^{r_1} \nabla f \cdot dr = f(r_1) - f(r_0) = |r_1|^2 - |r_0|^2 = (1 + 4 + 9).
\]

We conclude that \( I = 14 \). ◀

The line integral of conservative fields (Along a path.)

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \) along a straight line.

Solution: Consider the path \( C \) given by \( r(t) = (1, 2, 3) t \). Then \( r(0) = (0, 0, 0) \), and \( r(1) = (1, 2, 3) \). We now evaluate \( F = (2x, 2y, 2z) \) along \( r(t) \), that is, \( F(t) = (2t, 4t, 6t) \). Therefore,

\[
I = \int_{t_0}^{t_1} F(t) \cdot r'(t) \, dt = \int_0^1 (2t, 4t, 6t) \cdot (1, 2, 3) \, dt
\]

\[
I = \int_0^1 (2t + 8t + 18t) \, dt = \int_0^1 28t \, dt = 28 \left( \frac{t^2}{2} \right)_0^1.
\]

We conclude that \( I = 14 \). ◀
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Finding the potential of a gradient field

**Theorem (Characterization of gradient fields)**

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ on a simply connected domain $D \subset \mathbb{R}^3$ is a gradient field iff hold

$$
\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.
$$

**Proof:** Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is a gradient field, there exists a scalar field $f$ such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$
F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j F_i.
$$

(The statement ($\Leftarrow$) is more complicated to prove.)
Finding the potential of a gradient field

Example
Show that the field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \) is a gradient field.

Solution: We need to show that the equations in the Theorem above hold, that is

\[
\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.
\]

with \( x_1 = x, \ x_2 = y, \) and \( x_3 = z. \) This is the case, since

\[
\partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x, \\
\partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z, \\
\partial_3 F_1 = 0, \quad \partial_1 F_3 = 0.
\]

\[\triangleleft\]

Finding the potential of a gradient field

Example
Find the potential of the gradient field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle. \)

Solution: We know there exists a scalar function \( f \) solution of

\[
\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.
\]

\[
f = \int 2xy \, dx + g(y, z) \quad \Rightarrow \quad f = x^2y + g(y, z).
\]

\[
\partial_y f = x^2 + \partial_y g(y, z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y, z) = -z^2.
\]

\[
g(y, z) = -\int z^2 \, dy + h(z) = -z^2y + h(z) \quad \Rightarrow \quad f = x^2y - z^2y + h(z).
\]

\[
\partial_z f = -2zy + \partial_z h(z) = -2yz \quad \Rightarrow \quad \partial_z h(z) = 0 \quad \Rightarrow \quad f = (x^2 - z^2)y + c_0.
\]

\[\triangleleft\]
Conservative fields and potential functions. (Sect. 16.3)

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- **Comments on exact differential forms.**

Comments on exact differential forms

**Notation:** We call a *differential form* to the integrand in a line integral for a smooth field \( \mathbf{F} \), that is,

\[
\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.
\]

**Remark:** A differential form is a quantity that can be integrated along a path.

**Definition**

A differential form \( \mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz \) is called *exact* iff there exists a scalar function \( f \) such that

\[
F_x dx + F_y dy + F_z dz = \partial_x f dx + \partial_y f dy + \partial_z f dz.
\]

**Remarks:**

- A differential form \( \mathbf{F} \cdot d\mathbf{r} \) is exact iff \( \mathbf{F} = \nabla f \).
- In this context an exact differential form is nothing else than another name for a gradient field.
Comments on exact differential forms

Example
Show that the differential form given below is exact, where
\[ \mathbf{F} \cdot d\mathbf{r} = 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz. \]

Solution: We need to do the same calculation we did above:
Writing \( \mathbf{F} \cdot d\mathbf{r} = F_1 \, dx_1 + F_2 \, dx_2 + F_3 \, dx_3 \), show that
\[ \partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1. \]
with \( x_1 = x, \ x_2 = y, \) and \( x_3 = z. \) We showed that this is the case, since
\[ \partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x, \]
\[ \partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z, \]
\[ \partial_3 F_1 = 0, \quad \partial_1 F_3 = 0. \]
So, there exists \( f \) such that \( \mathbf{F} \cdot d\mathbf{r} = \nabla f \cdot d\mathbf{r}. \)