Integrals of vector fields. (Sect. 16.2)

- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - The flow of a fluid along a curve.
- The flux across a plane curve.

Vector fields on a plane and in space

Definition
A vector field on a plane or in space is a vector-valued function \( \mathbf{F}: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), respectively.

Examples from physics:
- Electric and magnetic fields.
- The gravitational field of the Earth.
- The velocity field in a fluid or gas.
- The variation of temperature in a room. (Gradient field.)

Magnetic field of a small magnet.
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**The gradient field of a scalar-valued function**

**Remark:**
- Given a scalar-valued function \( f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}, \) with \( n = 2, 3, \) its gradient vector, \( \nabla f = (\partial_x f, \partial_y f) \) or \( \nabla f = (\partial_x f, \partial_y f, \partial_z f), \) respectively, is a vector field in a plane or in space.

**Example**
Find and sketch a graph of the gradient field of the function \( f(x, y) = x^2 + y^2. \)

**Solution:** We know the graph of \( f \) is a paraboloid. The gradient field is \( \nabla f = (2x, 2y). \)
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The line integral of a vector field along a curve

**Definition**

The line integral of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s) \, ds$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points $t_0$, $t_1$.

**Example**

**Remark:** It is common the notation $\hat{\mathbf{r}}' = \mathbf{T}$, since $\mathbf{T}$ is tangent to the curve and unit, since $s$ is the curve arc-length parameter.
Line integrals in space

**Theorem (General parametrization formula)**

The line integral of a continuous function \( \mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3 \) along a differentiable curve \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) can be written as

\[
\int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s) \, ds = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,
\]

where \( \hat{\mathbf{r}}(s) \) is the arc length parametrization of the function \( \mathbf{r} \), and \( s(t_0) = s_0, s(t_1) = s_1 \) are the arc lengths at the points \( t_0, t_1 \).

**Proof:** Recall the curve arc-length function

\[ s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| \, d\tau. \]

Then \( ds = |\mathbf{r}'(t)| \, dt \). Also, \( \hat{\mathbf{r}}(s(t)) = \mathbf{r}(t) \). And finally

\[
\hat{\mathbf{r}}'(s) = \frac{d\hat{\mathbf{r}}}{ds}(s) = \frac{dr}{dt}(t) \frac{dt}{ds} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \implies \hat{\mathbf{r}}'(s) \, ds = \mathbf{r}'(t) \, dt.
\]

This substitution provides the equation in the Theorem. □

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Work done by a force on a particle

Definition
If the vector valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the work done by the force on the particle.

Example

A mass $m$ projectile near the Earth surface.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leq 0$ in the first half of the trajectory, and $W \geq 0$ on the second half.

Work done by a force on a particle

Example

Find the work done by the force $\mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$, $t \in [0, 1]$.

Solution:
First: Evaluate $\mathbf{F}$ along $\mathbf{r}$. This is: $\mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1 \rangle$.

Second: Compute $\mathbf{r}'(t)$. This is: $\mathbf{r}'(t) = \langle 1, 2t, 4t^3 \rangle$.

Third: Integrate the dot product $\mathbf{F}(t) \cdot \mathbf{r}'(t)$.

$$W = \int_0^1 [(3t^2 - 3t) + (6t^5) + (4t^3)] \, dt$$
$$= \left( t^3 - \frac{3}{2} t^2 + t^6 + t^4 \right) \bigg|_0^1 = 1 - \frac{3}{2} + 1 + 1.$$

So, $W = 3 - \frac{3}{2}$. We conclude: The work done is $W = \frac{3}{2}$. \(\triangleq\)
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The flow of a fluid along a curve

**Definition**

In the case that the vector field \( \mathbf{v} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), is the velocity field of a flow and \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) is any smooth curve, then the line integral

\[
F = \int_C \mathbf{v} \cdot d\mathbf{r},
\]

is called a flow integral. If the curve is a closed loop, the flow integral is called the circulation of the fluid around the loop.

**Example**

- The flow of a viscous fluid in a pipe is maximal along a line through the center of the pipe.
- The flow vanishes on any curve perpendicular to the section of the pipe.
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( r_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( r_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: \( F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2. \)

The first term is given by:

\[
\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^\pi \mathbf{v}(t) \cdot r_1'(t) \, dt.
\]

\( \mathbf{v}(t) = \langle -a \sin(t), a \cos(t) \rangle, \quad r_1'(t) = \langle -a \sin(t), a \cos(t) \rangle. \)

\[
\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^\pi a^2 [\sin^2(t) + \cos^2(t)] \, dt \iff \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2.
\]

The second term is given by:

\[
\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^a \mathbf{v}(t) \cdot r_2'(t) \, dt,
\]

\( \mathbf{v}(t) = \langle 0, t \rangle, \quad r_2'(t) = \langle 1, 0 \rangle. \)

\[
\mathbf{v}(t) \cdot r_2'(t) = 0 \iff \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = 0.
\]

Since \( \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2 \), we conclude: \( F = \pi a^2. \)

\[\triangleq\]
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**The flux across a plane curve**

**Definition**

The flux of a vector field \( \mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \rightarrow \{z = 0\} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow \{z = 0\} \subset \mathbb{R}^3 \) is given by

\[
\Phi = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is the curve outer unit normal vector in the plane \( \{z = 0\} \).

**Example**

**Remarks:**

- \( \mathbf{F} \) is defined on \( \{z = 0\} \).
- The loop \( C \) lies on \( \{z = 0\} \).
- Simple formula for \( \mathbf{n} \)? Yes.

\[
\mathbf{n} = \frac{1}{|r'|} \langle y'(t), -x'(t), 0 \rangle.
\]
The flux across a plane curve

Theorem (Counterclockwise loops.)

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} [F_x y'(t) - F_y x'(t)] \, dt.
\]

Proof:

Remarks: Since \( C \) is counterclockwise traversed, \( \mathbf{n} = \mathbf{u} \times \mathbf{k} \), where \( \mathbf{u} = \mathbf{r}'/|\mathbf{r}'| \).

\[
\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|} \langle x'(t), y'(t), 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.
\]

\[
\mathbf{n} = \frac{1}{|\mathbf{r}'|} \begin{vmatrix} i & j & k \\ x' & y' & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle.
\]
The flux across a plane curve

Example
Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)
Along \( C_1 \) we have: \( \mathbf{F}_1(t) = \langle -a \sin(t), a \cos(t), 0 \rangle \) and \( x'(t) = -a \sin(t), \ y'(t) = a \cos(t) \).

Therefore,
\[
F_{1x}(t) y'(t) - F_{1y}(t) x'(t) = -a^2 \sin(t) \cos(t) + a^2 \sin(t) \cos(t) = 0.
\]
Hence: \( \int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = 0. \)

The flux across a plane curve

Example
Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)
Along \( C_2 \) we have: \( \mathbf{F}_2(t) = \langle 0, t, 0 \rangle \) and \( x'(t) = 1, \ y'(t) = 0 \). So,
\[
F_{2x}(t) y'(t) - F_{2y}(t) x'(t) = 0 - t \quad \Rightarrow \quad \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{-a}^{a} -t \, dt,
\]
\[
\int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = \left. \left( \frac{t^2}{2} \right) \right|_{-a}^{a} \quad \Rightarrow \quad \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = 0.
\]
We conclude: \( \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = 0. \)
\[\triangleleft\]