Integrals of vector fields. (Sect. 16.2)

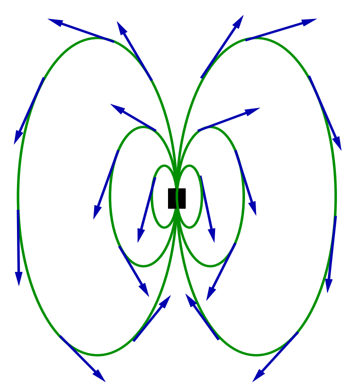
- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - The flow of a fluid along a curve.
- The flux across a plane curve.

Vector fields on a plane and in space

Definition
A *vector field* on a plane or in space is a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, respectively.

Examples from physics:

- Electric and magnetic fields.
- The gravitational field of the Earth.
- The velocity field in a fluid or gas.
- The variation of temperature in a room. (Gradient field.)



Magnetic field of a small magnet.

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The gradient field of a scalar-valued function

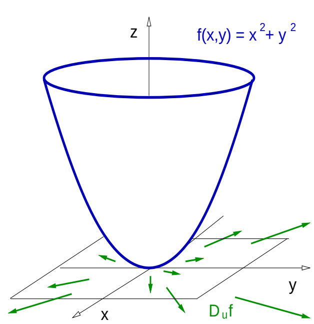
Remark:

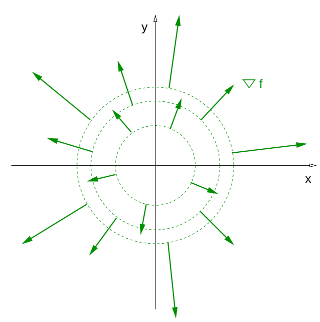
- Given a scalar-valued function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3$, its gradient vector, $\nabla f = \langle \partial_x f, \partial_y f \rangle$ or $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$, respectively, is a vector field in a plane or in space.

Example

Find and sketch a graph of the gradient field of the function $f(x, y) = x^2 + y^2$.

Solution: We know the graph of f is a paraboloid. The gradient field is $\nabla f = \langle 2x, 2y \rangle$.





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The line integral of a vector field along a curve

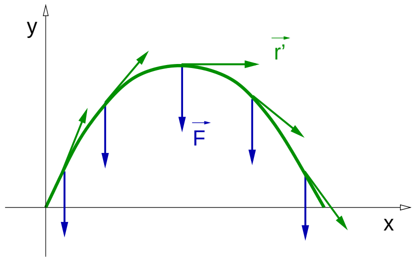
Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s)\, ds$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points t_0, t_1.

Example



Remark: It is common the notation

$$\hat{\mathbf{r}}' = \mathbf{T},$$

since $\mathbf{T}$ is tangent to the curve and unit, since s is the curve arc-length parameter.

Line integrals in space

Theorem (General parametrization formula)

The line integral of a continuous function $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ *along a differentiable curve* $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ *can be written as*

$$\int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s)\, ds = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

where $\hat{\mathbf{r}}(s)$ *is the arc length parametrization of the function* $\mathbf{r}$, *and* $s(t_0) = s_0$, $s(t_1) = s_1$ *are the arc lengths at the points* t_0, t_1.

Proof: Recall the curve arc-length function $s(t) = \displaystyle\int_{t_0}^{t} |\mathbf{r}'(\tau)|\, d\tau$.

Then $ds = |\mathbf{r}'(t)|\, dt$. Also, $\hat{\mathbf{r}}(s(t)) = \mathbf{r}(t)$. And finally

$$\hat{\mathbf{r}}'(s) = \frac{d\hat{\mathbf{r}}}{ds}(s) = \frac{d\mathbf{r}}{dt}(t)\,\frac{dt}{ds} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \;\Rightarrow\; \hat{\mathbf{r}}'(s)\, ds = \mathbf{r}'(t)\, dt.$$

This substitution provides the equation in the Theorem. $\square$

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Work done by a force on a particle

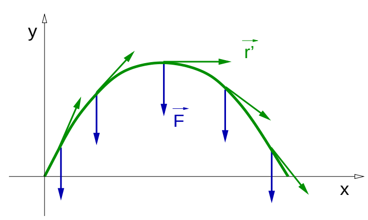
Definition

If the vector valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

Example



A mass m projectile near the Earth surface.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leqslant 0$ in the first half of the trajectory, and $W \geqslant 0$ on the second half.

Work done by a force on a particle

Example

Find the work done by the force $\mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$, $t \in [0, 1]$.

Solution:

First: Evaluate $\mathbf{F}$ along $\mathbf{r}$. This is: $\mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1 \rangle$.

Second: Compute $\mathbf{r}'(t)$. This is: $\mathbf{r}'(t) = \langle 1, 2t, 4t^3 \rangle$.

Third: Integrate the dot product $\mathbf{F}(t) \cdot \mathbf{r}'(t)$.

$$W = \int_0^1 \left[(3t^2 - 3t) + (6t^5) + (4t^3) \right] dt$$
$$= \left(t^3 - \frac{3}{2}t^2 + t^6 + t^4 \right)\Big|_0^1 = 1 - \frac{3}{2} + 1 + 1.$$

So, $W = 3 - \dfrac{3}{2}$. We conclude: The work done is $W = \dfrac{3}{2}$. $\triangleleft$

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The flow of a fluid along a curve

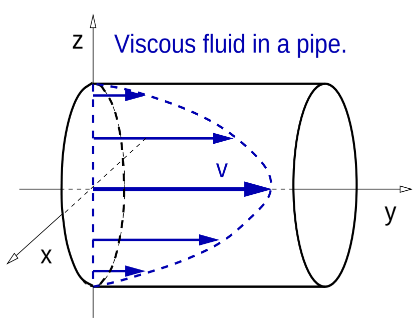
Definition

In the case that the vector field $\mathbf{v} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is the velocity field of a flow and $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is any smooth curve, then the line integral

$$F = \int_C \mathbf{v} \cdot d\mathbf{r},$$

is called a *flow integral*. If the curve is a closed loop, the flow integral is called the *circulation* of the fluid around the loop.

Example



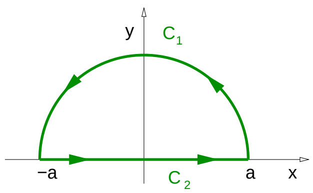
- The flow of a viscous fluid in a pipe is maximal along a line through the center of the pipe.
- The flow vanishes on any curve perpendicular to the section of the pipe.

The flow of a fluid along a curve

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2.$



The first term is given by:

$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^{\pi} \mathbf{v}(t) \cdot \mathbf{r}_1'(t)\, dt.$$

$$\mathbf{v}(t) = \langle -a\sin(t), a\cos(t) \rangle,$$

$$\mathbf{r}_1'(t) = \langle -a\sin(t), a\cos(t) \rangle.$$

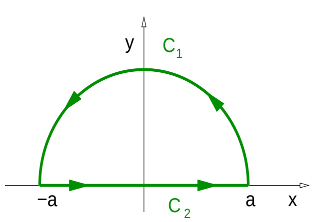
$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^{\pi} a^2 \big[\sin^2(t) + \cos^2(t)\big]\, dt \quad \Rightarrow \quad \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2.$$

The flow of a fluid along a curve

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2.$



The second term is given by:

$$\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}_2'(t)\, dt,$$

$$\mathbf{v}(t) = \langle 0, t \rangle, \quad \mathbf{r}_2'(t) = \langle 1, 0 \rangle.$$

$$\mathbf{v}(t) \cdot \mathbf{r}_2'(t) = 0 \quad \Rightarrow \quad \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = 0.$$

Since $\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2$, we conclude: $F = \pi a^2$. $\triangleleft$

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The flux across a plane curve

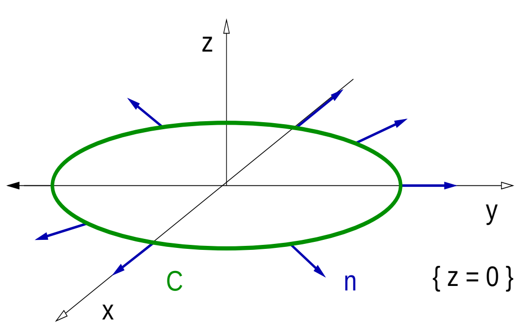
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the curve outer unit normal vector in the plane $\{z = 0\}$.

Example



Remarks:

- $\mathbf{F}$ is defined on $\{z = 0\}$.
- The loop C lies on $\{z = 0\}$.
- Simple formula for $\mathbf{n}$? Yes.

$$\mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle.$$

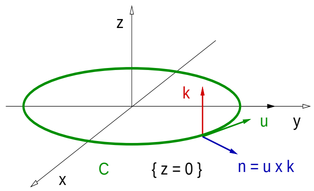
The flux across a plane curve

Theorem (Counterclockwise loops.)

The flux of a vector field $\mathbf{F} = \langle F_x(x,y), F_y(x,y), 0\rangle$ *along a closed, counterclockwise plane loop* $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$ *for* $t \in [t_0, t_1]$ *is given by*

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x\, y'(t) - F_y\, x'(t)\big]\, dt.$$

Proof:



Remarks: Since C is counterclockwise traversed, $\mathbf{n} = \mathbf{u}\times\mathbf{k}$, where $\mathbf{u} = \mathbf{r}'/|\mathbf{r}'|$.

$$\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|}\langle x'(t), y'(t), 0\rangle, \quad \mathbf{k} = \langle 0,0,1\rangle.$$

$$\mathbf{n} = \frac{1}{|\mathbf{r}'|}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \Rightarrow \quad \mathbf{n} = \frac{1}{|\mathbf{r}'|}\,\langle y'(t), -x'(t), 0\rangle.$$

The flux across a plane curve

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The flux of a vector field $\mathbf{F} = \langle F_x(x,y), F_y(x,y), 0\rangle$ *along a closed, counterclockwise plane loop* $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$ *for* $t \in [t_0, t_1]$ *is given by*

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x\, y'(t) - F_y\, x'(t)\big]\, dt.$$

Proof: Recall: $\mathbf{n} = \dfrac{1}{|\mathbf{r}'|}\,\langle y'(t), -x'(t), 0\rangle.$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_{t_0}^{t_1} \langle F_x, F_y, 0\rangle \cdot \langle y'(t), -x'(t), 0\rangle\, \frac{1}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x\, y'(t) - F_y\, x'(t)\big]\, dt. \qquad \square$$

The flux across a plane curve

Example

Find the flux of a field $\mathbf{F} = \langle -y, x, 0\rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t), 0\rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0\rangle$ for $t \in [-a, a]$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_1} \mathbf{F}_1\cdot\mathbf{n}_1\,ds + \int_{C_2} \mathbf{F}_2\cdot\mathbf{n}_2\,ds$

Along C_1 we have: $\mathbf{F}_1(t) = \langle -a\sin(t), a\cos(t), 0\rangle$ and

$$x'(t) = -a\sin(t), \quad y'(t) = a\cos(t).$$

Therefore,

$$F_{1x}(t)\,y'(t) - F_{1y}(t)\,x'(t) = -a^2\sin(t)\cos(t) + a^2\sin(t)\cos(t) = 0.$$

Hence: $\displaystyle\int_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds = 0.$

The flux across a plane curve

Example

Find the flux of a field $\mathbf{F} = \langle -y, x, 0\rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t), 0\rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0\rangle$ for $t \in [-a, a]$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_1} \mathbf{F}_1\cdot\mathbf{n}_1\,ds + \int_{C_2} \mathbf{F}_2\cdot\mathbf{n}_2\,ds$

Along C_2 we have: $\mathbf{F}_2(t) = \langle 0, t, 0\rangle$ and $x'(t) = 1$, $y'(t) = 0$. So,

$$F_{2x}(t)\,y'(t) - F_{2y}(t)\,x'(t) = 0 - t \quad \Rightarrow \quad \int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = \int_{-a}^{a} -t\,dt,$$

$$\int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = -\Big(\frac{t^2}{2}\Big|_{-a}^{a}\Big) \quad \Rightarrow \quad \int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = 0.$$

We conclude: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0.$ $\lhd$