Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- The addition of line integrals.
- Mass and center of mass of wires.

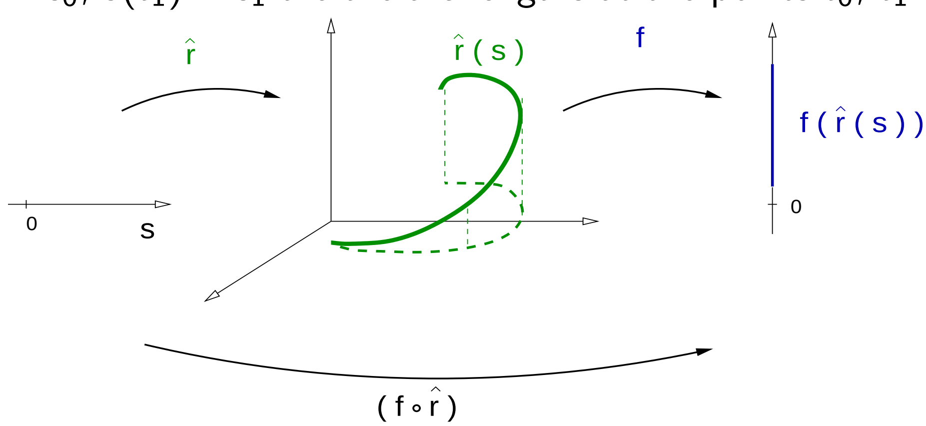
Line integrals in space

Definition

The *line integral* of a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along a curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C f\, ds = \int_{s_0}^{s_1} f(\hat{\mathbf{r}}(s))\, ds,$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points t_0, t_1.



Line integrals in space

Remarks:

- A line integral is an integral of a function along a curved path.
- The name *curved integrals* would be a better terminology.
- Line integrals originate in the early 1800 to study new physical situations. For example:
  (1) To describe the work done by a force along a path.
  (2) To solve problems involving fluid flows, electricity, magnetism.
- The line integral is *independent of the curve original parametrization*.
- In other words, given two different parametrization of the curve, with values $\mathbf{r}(t)$, and $\tilde{\mathbf{r}}(\tilde{t})$, then the value of the line integral is the same for both parametrizations.
- Because the line integral is computed using the curve arc-length parametrization, which is unique for every curve.

Line integrals in space

Theorem (General parametrization formula)

The line integral of a continuous function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along a differentiable curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ can be expressed as

$$\int_{s_0}^{s_1} f(\hat{\mathbf{r}}(s))\, ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt,$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points t_0, t_1.

Proof: Recall the curve arc-length function $s(t) = \displaystyle\int_{t_0}^{t} |\mathbf{r}'(\tau)|\, d\tau$.

Then $ds = |\mathbf{r}'(t)|\, dt$. Also, $\hat{\mathbf{r}}(s(t)) = \mathbf{r}(t)$. Then, the integration by substitution formula implies

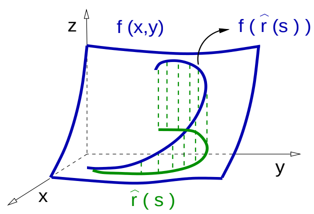
$$\int_{s_0}^{s_1} f\left[\hat{\mathbf{r}}(s(t))\right] ds = \int_{t_0}^{t_1} f\left(\mathbf{r}(t)\right) |\mathbf{r}'(t)|\, dt, \qquad \begin{aligned} s_0 &= s(t_0), \\ s_1 &= s(t_1). \end{aligned}$$

□

Line integrals in space

Remarks:

- When performing a line integral, the curve is always parametrized with its arc-length function.
- In this sense, a line integral is independent of the original parametrization of the curve.
- Line integrals can be defined on curves on the plane. In this case, the line integral is the area of the curtain under the graph of the function is the figure below.



The 2-dim line integral is an area, since the curve arc-length parametrization is used in the line integral computation.

Line integrals in space

Example

Evaluate the line integral of the function $f(x,y,z) = xy + y + z$ along the curve $\mathbf{r}(t) = \langle 2t, t, 2-2t \rangle$ in the interval $t \in [0,1]$.

Solution: ($\mathbf{r}$, straight line.) Recall: $\displaystyle\int_C f\,ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$

The derivative vector is $\mathbf{r}'(t) = \langle 2, 1, -2 \rangle$, therefore its magnitude is $|\mathbf{r}'(t)| = \sqrt{4+1+4} = 3$. The values of f along the curve are

$$f(\mathbf{r}(t)) = (2t)t + t + (2-2t) \quad \Rightarrow \quad f(\mathbf{r}(t)) = 2t^2 - t + 2.$$

$$\int_C f\,ds = \int_0^1 (2t^2 - t + 2)\,3\,dt = 3\Big[\Big(2\frac{t^3}{3} - \frac{t^2}{2} + 2t\Big)\Big|_0^1\Big].$$

$$\int_C f\,ds = 3\Big(\frac{2}{3} - \frac{1}{2} + 2\Big) = 2 - \frac{3}{2} + 6 \quad \Rightarrow \quad \int_C f\,ds = \frac{13}{2}. \;\triangleleft$$

Line integrals in space

Example

Evaluate the line integral of the function $f(x,y,z) = \sqrt{x^2 + z^2}$ along the curve $\mathbf{r}(t) = \langle 0, a\cos(t), a\sin(t)\rangle$, in $t \in [0, \pi/2]$.

Solution: ($\mathbf{r}$, half circle.)Recall: $\displaystyle\int_C f\,ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$

The derivative vector is $\mathbf{r}'(t) = \langle 0, -a\sin(t), a\cos(t)\rangle$, therefore its magnitude is $|\mathbf{r}'(t)| = \sqrt{a^2\sin^2(t) + a^2\cos^2(t)} = |a|$. The values of f along the curve are

$$f(\mathbf{r}(t)) = \sqrt{0 + a^2\sin^2(t)} \quad\Rightarrow\quad f(\mathbf{r}(t)) = |a||\sin(t)|.$$

$$\int_C f\,ds = \int_0^{\pi/2} |a|\sin(t)\,|a|\,dt = a^2\Big(-\cos(t)\Big|_0^{\pi/2}\Big).$$

$$\int_C f\,ds = a^2. \qquad \triangleleft$$

Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- **The addition of line integrals.**
- Mass and center of mass of wires.

The addition of line integrals

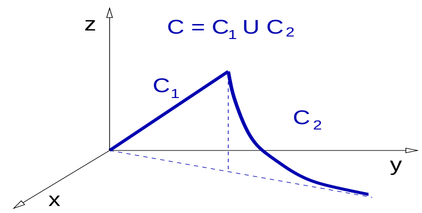
Theorem

If a curve $C \subset D$ in space is the union of the differentiable curves $C_1, \cdots, C_n$, then the line integral of a continuous function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along C satisfies

$$\int_C f\,ds = \int_{C_1} f\,ds + \cdots + \int_{C_n} f\,ds.$$

Remark:

This result is useful to compute line integral along piecewise differentiable curves.

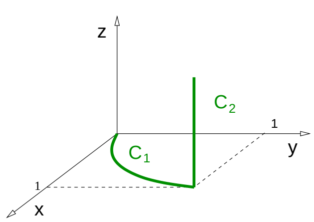


The addition of line integrals

Example

Evaluate the line integral of $f(x, y, z) = x + \sqrt{y} - z^2$ along the path $C = C_1 \cup C_2$, where C_1 is the image of $\mathbf{r}_1(t) = \langle t, t^2, 0\rangle$ for $t \in [0, 1]$, and C_2 is the image of $\mathbf{r}_2(t) = \langle 1, 1, t\rangle$ for $t \in [0, 1]$.

Solution:



$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds.$$

$$\mathbf{r}_1'(t) = \langle 1, 2t, 0\rangle \Rightarrow |\mathbf{r}_1'(t)| = \sqrt{1 + 4t^2}.$$

$$f(\mathbf{r}_1(t)) = t + t = 2t.$$

$$\int_{C_1} f\,ds = \int_0^1 2t\sqrt{1 + 4t^2}\,dt, \quad u = 1 + 4t^2,\ du = 8t\,dt.$$

$$\int_{C_1} f\,ds = \frac{1}{4}\int_1^5 u^{1/2}\,du = \frac{1}{4}\,\frac{2}{3}\left(u^{3/2}\Big|_1^5\right) \Rightarrow \int_{C_1} f\,ds = \frac{1}{6}(5\sqrt{5} - 1).$$

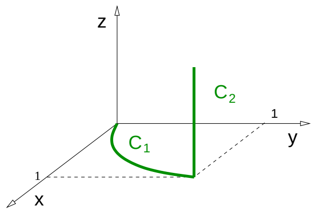
The addition of line integrals

Example

Evaluate the line integral of $f(x,y,z) = x + \sqrt{y} - z^2$ along the path $C = C_1 \cup C_2$, where C_1 is the image of $\mathbf{r}_1(t) = \langle t, t^2, 0\rangle$ for $t \in [0,1]$, and C_2 is the image of $\mathbf{r}_2(t) = \langle 1, 1, t\rangle$ for $t \in [0,1]$.

Solution:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds.$$



$$\mathbf{r}_2'(t) = \langle 0,0,1\rangle \Rightarrow |\mathbf{r}_2'(t)| = 1.$$

$$f(\mathbf{r}_2(t)) = 1 + 1 - t^2 = 2 - t^2.$$

$$\int_{C_2} f\,ds = \int_0^1 (2 - t^2)\,dt = 2\Big(t\Big|_0^1\Big) - \Big(\frac{t^3}{3}\Big|_0^1\Big) = 2 - \frac{1}{3} = \frac{5}{3}.$$

$$\int_C f\,ds = \frac{1}{6}(5\sqrt{5} - 1) + \frac{5}{3} \quad \Rightarrow \quad \int_{C_1} f\,ds = \frac{1}{6}(5\sqrt{5} + 9). \quad \triangleleft$$

Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- The addition of line integrals.
- **Mass and center of mass of wires.**

Mass and center of mass of wires

Remark:
The total mass, the center of mass, and the moments of inertia of wires with arbitrary shapes in space, given by a curve C and having a density function ρ, can be computed using line integrals.

- $M = \int_C \rho \, ds$;
- $\bar{x} = \dfrac{1}{M} \displaystyle\int_C x\rho \, ds, \quad \bar{y} = \dfrac{1}{M} \int_C y\rho \, ds, \quad \bar{z} = \dfrac{1}{M} \int_C z\rho \, ds$;
- $I_x = \dfrac{1}{M} \displaystyle\int_C (y^2 + z^2)\rho \, ds$,
- $I_y = \dfrac{1}{M} \displaystyle\int_C (x^2 + z^2)\rho \, ds$,
- $I_z = \dfrac{1}{M} \displaystyle\int_C (x^2 + y^2)\rho \, ds$.

Mass and center of mass of wires

Example
Find the moments of inertia of a wheel of radius R and density ρ_0.

Solution: We place the wheel at the center of the $z = 0$ plane. The curve for the wheel is $\mathbf{r}(t) = \langle R\cos(t), R\sin(t), 0\rangle$, $t \in [0, 2\pi]$. Therefore, $\mathbf{r}'(t) = \langle -R\sin(t), R\cos(t), 0\rangle$, hence $|\mathbf{r}'(t)| = R$. Recall: $I_x = \int_C (y^2 + z^2)\rho_0 \, ds$, $I_z = \int_C (x^2 + y^2)\rho_0 \, ds$.

$$I_x = \int_0^{2\pi} R^2 \sin^2(t)\rho_0 R \, dt = R^3\rho_0 \int_0^{2\pi} \frac{1}{2}\big[1 - \cos(2t)\big] \, dt$$

$$I_x = R^3\rho_0\Big[\pi - \frac{1}{4}\Big(\sin(2t)\Big|_0^{2\pi}\Big)\Big] \quad \Rightarrow \quad I_x = \pi R^3 \rho_0.$$

By symmetry, $I_x = I_y$. Finally,

$$I_z = \int_0^{2\pi} R^2 \rho_0 R \, dt \quad \Rightarrow \quad I_z = 2\pi R^3 \rho_0. \qquad \triangleleft$$