Cylindrical coordinates in space.

Definition
The **cylindrical coordinates** of a point \( P \in \mathbb{R}^3 \) is the ordered triple \((r, \theta, z)\) defined by the picture.

Remark: Cylindrical coordinates are just polar coordinates on the plane \( z = 0 \) together with the vertical coordinate \( z \).

Theorem (Cartesian-cylindrical transformations)
*The Cartesian coordinates of a point \( P = (r, \theta, z) \) are given by \( x = r \cos(\theta), \ y = r \sin(\theta), \) and \( z = z \).*

*The cylindrical coordinates of a point \( P = (x, y, z) \) in the first and fourth quadrant are \( r = \sqrt{x^2 + y^2}, \ \theta = \arctan(y/x), \) and \( z = z \).*
Integrals in cylindrical, spherical coordinates (Sect. 15.7)

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - **Spherical coordinates in space.**
  - Triple integral in spherical coordinates.

### Spherical coordinates in \( \mathbb{R}^3 \)

**Definition**
The spherical coordinates of a point \( P \in \mathbb{R}^3 \) is the ordered triple \((\rho, \phi, \theta)\) defined by the picture.

**Theorem (Cartesian-spherical transformations)**
*The Cartesian coordinates of \( P = (\rho, \phi, \theta) \) in the first quadrant are given by* \( x = \rho \sin(\phi) \cos(\theta) \), \( y = \rho \sin(\phi) \sin(\theta) \), and \( z = \rho \cos(\phi) \).*

*The spherical coordinates of \( P = (x, y, z) \) in the first quadrant are* \( \rho = \sqrt{x^2 + y^2 + z^2} \), \( \theta = \arctan\left(\frac{y}{x}\right) \), and \( \phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \).
Spherical coordinates in $\mathbb{R}^3$

Example

Use spherical coordinates to express region between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: $(x = \rho \sin(\phi) \cos(\theta), y = \rho \sin(\phi) \sin(\theta), z = \rho \cos(\phi).)$

The top surface is the sphere $\rho = 1$.
The bottom surface is the cone:

$$\rho \cos(\phi) = \sqrt{\rho^2 \sin^2(\phi)}$$
$$\cos(\phi) = \sin(\phi),$$

so the cone is $\phi = \frac{\pi}{4}$.

Hence: $R = \left\{ (\rho, \phi, \theta) : \theta \in [0, 2\pi], \phi \in \left[ 0, \frac{\pi}{4} \right], \rho \in [0, 1] \right\}$.

Integrals in cylindrical, spherical coordinates (Sect. 15.7)

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - Spherical coordinates in space.
  - **Triple integral in spherical coordinates.**
Triple integral in spherical coordinates

Theorem

If the function \( f : R \subset \mathbb{R}^3 \rightarrow \mathbb{R} \) is continuous, then the triple integral of function \( f \) in the region \( R \) can be expressed in spherical coordinates as follows,

\[
\int\int\int_R f \, dv = \int\int\int_R f(\rho, \phi, \theta) \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta.
\]

Remark:

- Spherical coordinates are useful when the integration region \( R \) is described in a simple way using spherical coordinates.
- Notice the extra factor \( \rho^2 \sin(\phi) \) on the right-hand side.

Example

Find the volume of a sphere of radius \( R \).

Solution: Sphere: \( S = \{ \theta \in [0, 2\pi], \phi \in [0, \pi], \rho \in [0, R]\} \).

\[
V = \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta,
\]

\[
V = \left[ \int_0^{2\pi} d\theta \right] \left[ \int_0^\pi \sin(\phi) \, d\phi \right] \left[ \int_0^R \rho^2 \, d\rho \right],
\]

\[
V = 2\pi \left[ -\cos(\phi) \right]_0^\pi \frac{R^3}{3},
\]

\[
V = 2\pi \left[ -\cos(\pi) + \cos(0) \right] \frac{R^3}{3};
\]

hence: \( V = \frac{4}{3} \pi R^3 \).
Triple integral in spherical coordinates

Example
Use spherical coordinates to find the volume below the sphere \( x^2 + y^2 + z^2 = 1 \) and above the cone \( z = \sqrt{x^2 + y^2} \).

Solution: \( R = \{ (\rho, \phi, \theta) : \theta \in [0, 2\pi], \phi \in [0, \pi/4], \rho \in [0, 1] \} \).
The calculation is simple, the region is a simple section of a sphere.

\[
V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta,
\]
\[
V = \left[ \int_0^{2\pi} d\theta \right] \left[ \int_0^{\pi/4} \sin(\phi) \, d\phi \right] \left[ \int_0^1 \rho^2 \, d\rho \right],
\]
\[
V = 2\pi \left[ -\cos(\phi) \right]_0^{\pi/4} \left( \frac{\rho^3}{3} \right)_0^1,
\]
\[
V = 2\pi \left[ -\frac{\sqrt{2}}{2} + 1 \right] \frac{1}{3} \Rightarrow V = \frac{\pi}{3} (2 - \sqrt{2}). \quad \triangleleft
\]

Triple integral in spherical coordinates

Example
Find the integral of \( f(x, y, z) = e^{(x^2 + y^2 + z^2)^{3/2}} \) in the region \( R = \{ x \geq 0, \ y \geq 0, \ z \geq 0, \ x^2 + y^2 + z^2 \leq 1 \} \) using spherical coordinates.

Solution: \( R = \{ \theta \in [0, \pi/2], \ \phi \in [0, \pi/2], \ \rho \in [0, 1] \} \). Hence,

\[
I = \int \int \int_R f \, dv = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 e^{\rho^3 \rho^2 \sin(\phi)} \, d\rho \, d\phi \, d\theta,
\]
\[
I = \left[ \int_0^{\pi/2} d\theta \right] \left[ \int_0^{\pi/2} \sin(\phi) \, d\phi \right] \left[ \int_0^1 e^{\rho^3 \rho^2} \, d\rho \right].
\]

Use substitution: \( u = \rho^3 \), hence \( du = 3\rho^2 \, d\rho \), so

\[
I = \frac{\pi}{2} \left[ -\cos(\phi) \right]_0^{\pi/2} \int_0^1 \frac{e^u}{3} \, du \Rightarrow \int \int \int_R f \, dv = \frac{\pi}{6} (e - 1). \quad \triangleleft
\]
Example
Change to spherical coordinates and compute the integral

\[ I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx. \]

Solution: \((x = \rho \sin(\phi) \cos(\theta), \ y = \rho \sin(\phi) \sin(\theta), \ z = \rho \cos(\phi)).\)

- Limits in \(x\): \(|x| \leq 2;\)
- Limits in \(y\): \(0 \leq y \leq \sqrt{4 - x^2}\), so the positive side of the disk \(x^2 + y^2 \leq 4.\)
- Limits in \(z\):
  \(0 \leq z \leq \sqrt{4 - x^2 - y^2}\), so a positive quarter of the ball \(x^2 + y^2 + z^2 \leq 4.\)

\[ \int_{0}^{\pi} \int_{0}^{\pi/2} \int_{0}^{2} \rho^2 \sin(\phi) \sin(\theta) (\rho^2 \sin(\phi)) \, d\rho \, d\phi \, d\theta. \]
Triple integral in spherical coordinates

Example
Change to spherical coordinates and compute the integral

\[ I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx. \]

Solution: \[ I = \int_{0}^{\pi} \int_{0}^{\pi/2} \int_{0}^{2} \rho^2 \sin(\phi) \sin(\theta) (\rho^2 \sin(\phi)) \, d\rho \, d\phi \, d\theta. \]

\[ I = \left[ \int_{0}^{\pi} \sin(\theta) \, d\theta \right] \left[ \int_{0}^{\pi/2} \sin^2(\phi) \, d\phi \right] \left[ \int_{0}^{2} \rho^4 \, d\rho \right], \]

\[ I = \left( -\cos(\theta) \right)_{0}^{\pi} \left[ \int_{0}^{\pi/2} \frac{1}{2} (1 - \cos(2\phi)) \, d\phi \right] \left( \rho^5 \right)_{0}^{2}, \]

\[ I = 2 \frac{1}{2} \left[ \left( \frac{\pi}{2} - 0 \right) - \frac{1}{2} \left( \sin(2\phi) \right)_{0}^{\pi/2} \right] \frac{2^5}{5} \Rightarrow I = \frac{2^4 \pi}{5}. \]

\[ \triangle \]

Triple integral in spherical coordinates

Example
Compute the integral \[ I = \int_{0}^{2\pi} \int_{0}^{\pi/3} \int_{\sec(\phi)}^{2} 3\rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta. \]

Solution: Recall: \( \sec(\phi) = 1/ \cos(\phi) \).

\[ I = 2\pi \int_{0}^{\pi/3} \left( \rho^3 \right)_{\sec(\phi)}^{2} \sin(\phi) \, d\phi, \]

\[ I = 2\pi \int_{0}^{\pi/3} \left( 2^3 - \frac{1}{\cos^3(\phi)} \right) \sin(\phi) \, d\phi \]

In the second term substitute: \( u = \cos(\phi), \, du = -\sin(\phi) \, d\phi \).

\[ I = 2\pi \left[ 2^3 \left( -\cos(\phi) \right)_{0}^{\pi/3} \right] + \int_{1}^{1/2} \frac{du}{u^3}. \]
Example

Compute the integral

\[ I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^{2} 3\rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta. \]

Solution: \[ I = 2\pi \left[ 2^3 \left( -\cos(\phi) \right) \bigg|_{\pi/3}^0 \right] + \int_1^{1/2} \frac{du}{u^3}. \]

\[ I = 2\pi \left[ 2^3 \left( -\frac{1}{2} + 1 \right) - \int_{1/2}^1 u^{-3} \, du \right] = 2\pi \left[ 4 - \left( \frac{u^{-2}}{-2} \bigg|_{1/2}^{1} \right) \right], \]

\[ I = 2\pi \left[ 4 + \frac{1}{2} \left( u^{-2} \bigg|_{1/2}^{1} \right) \right] = 2\pi \left[ 4 + \frac{1}{2} (1 - 2^2) \right] = 2\pi \left[ \frac{8}{2} - \frac{3}{2} \right] \]

We conclude: \( I = 5\pi. \)

Example

Use spherical coordinates to find the volume of the region outside the sphere \( \rho = 2 \cos(\phi) \) and inside the half sphere \( \rho = 2 \) with \( \phi \in [0, \pi/2] \).

Solution: First sketch the integration region.

- \( \rho = 2 \cos(\phi) \) is a sphere, since

\[ \rho^2 = 2\rho \cos(\phi) \Leftrightarrow x^2 + y^2 + z^2 = 2z \]

\[ x^2 + y^2 + (z - 1)^2 = 1. \]

- \( \rho = 2 \) is a sphere radius 2 and \( \phi \in [0, \pi/2] \) says we only consider the upper half of the sphere.
Example
Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2 \cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution:

\[ V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2 \cos(\phi)}^2 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta. \]

\[ V = 2\pi \int_0^{\pi/2} \left( \frac{\rho^3}{3} \right)_{2 \cos(\phi)} \sin(\phi) \, d\phi \]

\[ V = \frac{2\pi}{3} \int_0^{\pi/2} \left[ 8\sin(\phi) - 8\cos^3(\phi) \sin(\phi) \right] d\phi. \]

\[ V = \frac{16\pi}{3} \left[ \left( -\cos(\phi) \right)_{\pi/2}^0 \right] - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi) \, d\phi. \]

Introduce the substitution: $u = \cos(\phi)$, $du = -\sin(\phi) \, d\phi$.

\[ V = \frac{16\pi}{3} \left[ 1 + \int_1^0 u^3 \, du \right] = \frac{16\pi}{3} \left[ 1 + \left( \frac{u^4}{4} \right)_{0}^{1} \right] = \frac{16\pi}{3} \left( 1 - \frac{1}{4} \right). \]

\[ V = \frac{16\pi}{3} \cdot \frac{3}{4} \Rightarrow V = 4\pi. \]