Review for Exam 3

- Sections 15.1-15.5, 15.7.
- 50 minutes.
- 5, 6, problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Triple integral in spherical coordinates (Sect. 15.7)

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the half sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

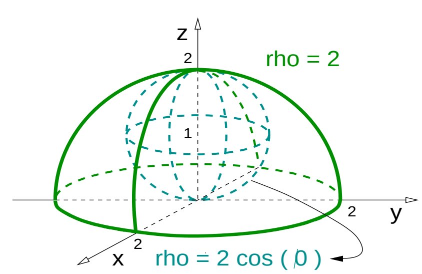
Solution: First sketch the integration region.

- $\rho = 2\cos(\phi)$ is a sphere, since

$$\rho^2 = 2\rho\cos(\phi) \Leftrightarrow x^2+y^2+z^2 = 2z$$

$$x^2 + y^2 + (z-1)^2 = 1.$$

- $\rho = 2$ is a sphere radius 2 and $\phi \in [0, \pi/2]$ says we only consider the upper half of the sphere.



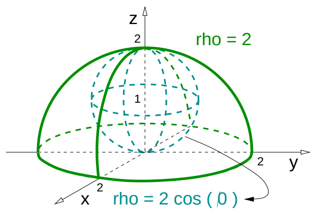
Triple integral in spherical coordinates (Sect. 15.7)

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution:

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos(\phi)}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$



$$V = 2\pi \int_0^{\pi/2} \Big(\frac{\rho^3}{3}\Big|_{2\cos(\phi)}^{2} \Big) \sin(\phi)\, d\phi$$

$$V = \frac{2\pi}{3} \int_0^{\pi/2} \Big[8\sin(\phi) - 8\cos^3(\phi)\, \sin(\phi) \Big]\, d\phi.$$

$$V = \frac{16\pi}{3} \Big[\Big(-\cos(\phi)\Big|_0^{\pi/2} \Big) - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi \Big].$$

Triple integral in spherical coordinates (Sect. 15.7)

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution: $V = \frac{16\pi}{3} \Big[\Big(-\cos(\phi)\Big|_0^{\pi/2} \Big) - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi \Big].$

Introduce the substitution: $u = \cos(\phi)$, $du = -\sin(\phi)\, d\phi$.

$$V = \frac{16\pi}{3} \Big[1 + \int_1^0 u^3\, du \Big] = \frac{16\pi}{3} \Big[1 + \Big(\frac{u^4}{4}\Big|_1^0 \Big) \Big] = \frac{16\pi}{3} \Big(1 - \frac{1}{4} \Big).$$

$$V = \frac{16\pi}{3}\,\frac{3}{4} \quad \Rightarrow \quad V = 4\pi. \qquad \triangleleft$$

Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume in the $z \geqslant 0$ region of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

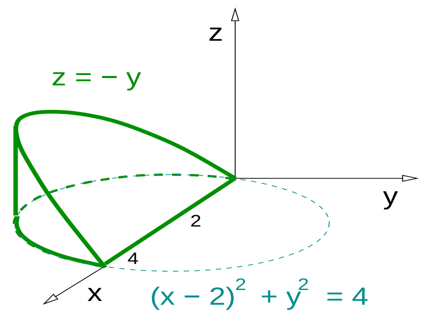
Solution: First sketch the integration region.

- $(x-2)^2 + y^2 = 4$ is a circle in the xy-plane, since

$$x^2 + y^2 = 4x \Leftrightarrow r^2 = 4r\cos(\theta)$$

$$r = 4\cos(\theta).$$

- Since $0 \leqslant z \leqslant -y$, the integration region is on the $y \leqslant 0$ part of the $z = 0$ plane.

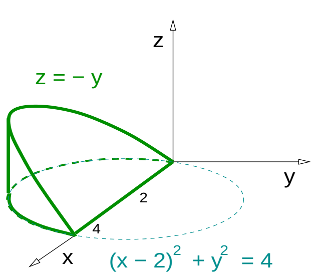


Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume in the $z \geqslant 0$ region of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution:



$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \int_0^{-r\sin(\theta)} r\,dz\,dr\,d\theta.$$

$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \big[-r\sin(\theta) - 0\big]\,r\,dr\,d\theta$$

$$V = -\int_{3\pi/2}^{2\pi} \Big(\frac{r^3}{3}\Big|_0^{4\cos(\theta)}\Big)\sin(\theta)\,d\theta.$$

$$V = -\int_{3\pi/2}^{2\pi} \frac{4^3}{3}\cos^3(\theta)\sin(\theta)\,d\theta.$$

Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution: $V = -\int_{3\pi/2}^{2\pi} \frac{4^3}{3} \cos^3(\theta) \sin(\theta)\, d\theta.$

Introduce the substitution: $u = \cos(\theta)$, $du = -\sin(\theta)\, d\theta$;

$$V = \frac{4^3}{3} \int_0^1 u^3\, du = \frac{4^3}{3} \Big(\frac{u^4}{4}\Big|_0^1 \Big) = \frac{4^3}{3}\, \frac{1}{4}.$$

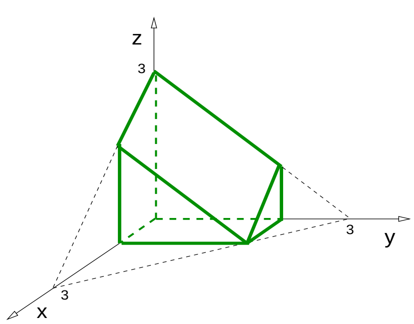
We conclude: $V = \frac{16}{3}$. ◁

Triple integral in Cartesian coordinates (Sect. 15.5)

Example

Find the volume of a parallelepiped whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 2$ and $0 \leqslant x \leqslant 1$, while the top side lies in the plane $x + y + z = 3$.

Solution:



$$V = \int_0^1 \int_0^2 \int_0^{3-x-y} dz\, dy\, dx,$$

$$V = \int_0^1 \int_0^2 (3 - x - y)\, dy\, dx,$$

$$V = \int_0^1 \Big[(3-x)\Big(y\Big|_0^2\Big) - \frac{1}{2}\Big(y^2\Big|_0^2\Big) \Big]\, dx,$$

$$V = \int_0^1 \Big[2(3-x) - \frac{4}{2} \Big]\, dx.$$

$$V = \int_0^1 (4 - 2x)\, dx = \Big[4\Big(x\Big|_0^1\Big) - \Big(x^2\Big|_0^1\Big) \Big] = 4 - 1 \;\Rightarrow\; V = 3.$$ ◁

Triple integral in Cartesian coordinates (Sect. 15.5)

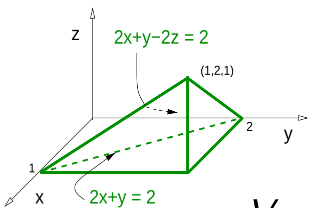
Example

Find the volume of the region in the first octant below the plane $2x + y - 2z = 2$ and $x \leqslant 1$, $y \leqslant 2$.

Solution: First sketch the integration region.

The plane contains the points $(1, 0, 0)$, $(0, 2, 0)$, $(1, 2, 1)$.

We choose the order $dz\, dy\, dx$. The integral is



$$V = \int_0^1 \int_{2-2x}^2 \int_0^{-1+x+y/2} dz\, dy\, dx.$$

$$V = \int_0^1 \int_{2-2x}^2 \Big[(-1 + x) + \frac{y}{2}\Big] dy\, dx,$$

$$V = \int_0^1 \Big[-(1-x)[2 - 2(1-x)] + \frac{1}{4}[4 - 4(1-x)^2]\Big]\, dx.$$

Triple integral in Cartesian coordinates (Sect. 15.5)

Example

Find the volume of the region in the first octant below the plane $2x + y - 2z = 2$ and $x \leqslant 1$, $y \leqslant 2$.

Solution: $V = \int_0^1 \Big[-(1-x)[2 - 2(1-x)] + \frac{1}{4}[4 - 4(1-x)^2]\Big]\, dx.$

$$V = \int_0^1 \big[-2(1-x) + 2(1-x)^2 + 1 - (1-x)^2\big]\, dx,$$

$$V = \int_0^1 \big[-1 + 2x + (1-x)^2\big]\, dx = \int_0^1 \big[-1 + 2x + 1 + x^2 - 2x\big]\, dx$$

$$V = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 \quad \Rightarrow \quad V = \frac{1}{3}. \qquad \lhd$$

Double integrals in polar coordinates. (Sect. 15.4)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

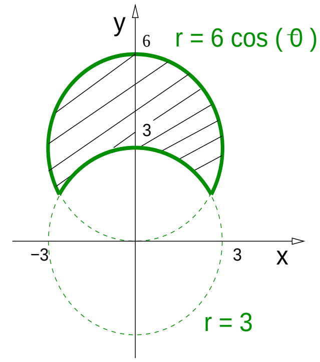
Solution: First sketch the integration region.

- $r = 6\sin(\theta)$ is a circle, since

$$r^2 = 6r\sin(\theta) \Leftrightarrow x^2 + y^2 = 6y$$

$$x^2 + (y-3)^2 = 3^2.$$

- The other curve is a circle $r = 3$ centered at the origin.



The condition $3 = r = 6\sin(\theta)$ determines the range in θ. Since $\sin(\theta) = 1/2$, we get $\theta_1 = 5\pi/6$ and $\theta_0 = \pi/6$.

Double integrals in polar coordinates. (Sect. 15.4)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

Solution: Recall: $\theta \in [\pi/6, 5\pi/6]$. The area is

$$A = \int_{\pi/6}^{5\pi/6} \int_3^{6\sin(\theta)} r dr\, d\theta = \int_{\pi/6}^{5\pi/6} \Big(\frac{r^2}{2}\Big|_3^{6\sin(\theta)} \Big)\, d\theta$$

$$A = \int_{\pi/6}^{5\pi/6} \Big[\frac{6^2}{2}\sin^2(\theta) - \frac{3^2}{2} \Big]\, d\theta = \int_{\pi/6}^{5\pi/6} \Big[\frac{6^2}{2^2}\big(1 - \cos(2\theta)\big) - \frac{3^2}{2} \Big]\, d\theta$$

$$A = 3^2\Big(\frac{5\pi}{6} - \frac{\pi}{6}\Big) - \frac{3^2}{2}\Big(\sin(2\theta)\Big|_{\pi/6}^{5\pi/6}\Big) - \frac{3^2}{2}\Big(\frac{5\pi}{6} - \frac{\pi}{6}\Big).$$

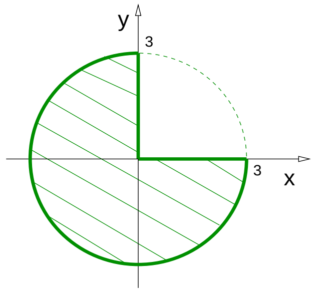
$$A = 6\pi - 3\pi - \frac{3^2}{2}\Big(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\Big), \text{ hence } A = 3\pi + 9\sqrt{3}/2. \qquad \triangleleft$$

Double integrals in polar coordinates. (Sect. 15.4)

Example

Use polar coordinate to compute the integral $\overline{y} = \dfrac{1}{A}\iint_R y\,dx\,dy$, where A is the area of the region in the plane given by the disk $x^2 + y^2 \leqslant 9$ minus the first quadrant.

Solution: First sketch the integration region.



Therefore, $A = \pi R^2 (3/4)$, with $R = 3$. We use polar coordinates to compute $\overline{y}$.

$$\overline{y} = \frac{4}{27\pi}\int_{\pi/2}^{2\pi}\int_0^3 r\sin(\theta)\,r dr\,d\theta.$$

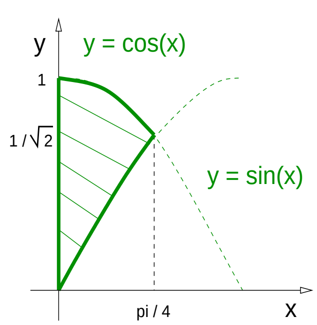
$$\overline{y} = \frac{4}{27\pi}\Big(-\cos(\theta)\Big|_{\pi/2}^{2\pi}\Big)\Big(\frac{r^3}{3}\Big|_0^3\Big) = \frac{4}{27\pi}(-1)(9) \;\Rightarrow\; \overline{y} = -\frac{4}{3\pi}. \qquad \triangleleft$$

Double integrals in Cartesian coordinates (Section 15.3)

Example

Switch the integration order in $I = \displaystyle\int_0^{\pi/4}\int_{\sin(x)}^{\cos(x)} dy\,dx.$

Solution: We first draw the integration region.



Divide the region at $y = \dfrac{1}{\sqrt{2}}$.

$$I = \int_0^{1/\sqrt{2}}\int_0^{\arcsin(y)} dx\,dy + \int_{1/\sqrt{2}}^{1}\int_0^{\arccos(y)} dx\,dy.$$

So, $I = \displaystyle\int_0^{1/\sqrt{2}}\int_0^{\arcsin(y)} dx\,dy + \int_{1/\sqrt{2}}^{1}\int_0^{\arccos(y)} dx\,dy.$ $\triangleleft$

Double integrals in Cartesian coordinates (Section 15.2)

Example

Switch the integration order in $I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx.$

Solution:

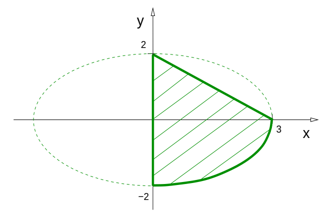
We first draw the integration region. Start with the outer limits.

$x \in [0,3]$.

$y \leqslant 2 - 2x/3$ and $y \geqslant 2\sqrt{1 - \frac{x^2}{3^2}}$.

The lower limit is part of the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$$

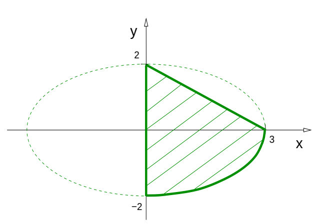


Double integrals in Cartesian coordinates (Section 15.2)

Example

Switch the integration order in $I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx.$

Solution:



Split the integral at $y = 0$.
In $y \in [-2, 0]$, holds $0 \leqslant x$.
The upper limit comes from
$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$,
so, $x = +3\sqrt{1 - \frac{y^2}{2^2}}$.

In $y \in [0,2]$, holds $0 \leqslant x$. The upper limit comes from $y = 2\Big(1 - \frac{x}{3}\Big)$, that is, $x = 3\Big(1 - \frac{y}{2}\Big)$. We then conclude:

$$I = \int_{-2}^{0} \int_0^{3\sqrt{1-\frac{y^2}{2^2}}} f(x,y)\,dx\,dy + \int_0^2 \int_0^{3(1-\frac{y}{2})} f(x,y)\,dx\,dy.$$

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