Review for Exam 3

- Sections 14.7, 15.1-15.5, 15.7.
- 50 minutes.
- 5, 6, problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume in the $z \geqslant 0$ region of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

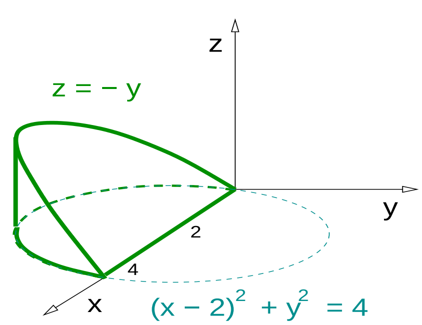
Solution: First sketch the integration region.

- $(x-2)^2 + y^2 = 4$ is a circle in the xy-plane, since

$$x^2 + y^2 = 4x \Leftrightarrow r^2 = 4r\cos(\theta)$$

$$r = 4\cos(\theta).$$

- Since $0 \leqslant z \leqslant -y$, the integration region is on the $y \leqslant 0$ part of the $z = 0$ plane.

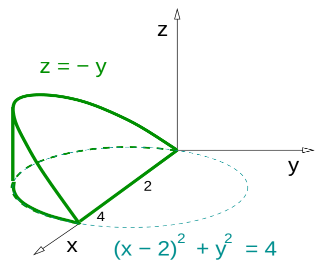


Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume in the $z \geqslant 0$ region of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution:



$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \int_0^{-r\sin(\theta)} r \, dz \, dr \, d\theta.$$

$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \big[-r\sin(\theta) - 0\big] \, r \, dr \, d\theta$$

$$V = -\int_{3\pi/2}^{2\pi} \Big(\frac{r^3}{3}\Big|_0^{4\cos(\theta)} \Big) \sin(\theta) \, d\theta.$$

$$V = -\int_{3\pi/2}^{2\pi} \frac{4^3}{3} \cos^3(\theta) \sin(\theta) \, d\theta.$$

Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution: $V = -\displaystyle\int_{3\pi/2}^{2\pi} \frac{4^3}{3} \cos^3(\theta) \sin(\theta) \, d\theta.$

Introduce the substitution: $u = \cos(\theta)$, $du = -\sin(\theta)\, d\theta$;

$$V = \frac{4^3}{3} \int_0^1 u^3 \, du = \frac{4^3}{3} \Big(\frac{u^4}{4}\Big|_0^1 \Big) = \frac{4^3}{3} \, \frac{1}{4}.$$

We conclude: $V = \dfrac{16}{3}$. ◁

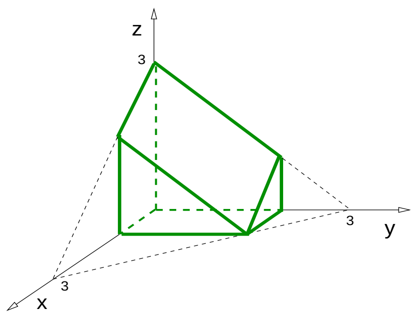
Triple integral in Cartesian coordinates (Sect. 15.5)

Example

Find the volume of a parallelepiped whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 2$ and $0 \leqslant x \leqslant 1$, while the top side lies in the plane $x + y + z = 3$.

Solution:

$$V = \int_0^1 \int_0^2 \int_0^{3-x-y} dz\, dy\, dx,$$



$$V = \int_0^1 \int_0^2 (3 - x - y)\, dy\, dx,$$

$$V = \int_0^1 \left[(3 - x)\Big(y\Big|_0^2 \Big) - \frac{1}{2}\Big(y^2\Big|_0^2 \Big) \right] dx,$$

$$V = \int_0^1 \left[2(3 - x) - \frac{4}{2} \right] dx.$$

$$V = \int_0^1 (4 - 2x)\, dx = \left[4\Big(x\Big|_0^1 \Big) - \Big(x^2\Big|_0^1 \Big) \right] = 4 - 1 \;\Rightarrow\; V = 3.$$

$\triangleleft$

Triple integral in Cartesian coordinates (Sect. 15.5)

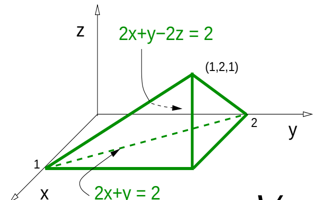
Example

Find the volume of the region in the first octant below the plane $2x + y - 2z = 2$ and $x \leqslant 1$, $y \leqslant 2$.

Solution: First sketch the integration region.

The plane contains the points $(1, 0, 0)$, $(0, 2, 0)$, $(1, 2, 1)$.

We choose the order $dz\, dy\, dx$. The integral is



$$V = \int_0^1 \int_{2-2x}^2 \int_0^{-1+x+y/2} dz\, dy\, dx.$$

$$V = \int_0^1 \int_{2-2x}^2 \left[(-1 + x) + \frac{y}{2} \right] dy\, dx,$$

$$V = \int_0^1 \left[-(1 - x)[2 - 2(1 - x)] + \frac{1}{4}[4 - 4(1 - x)^2] \right] dx.$$

Triple integral in Cartesian coordinates (Sect. 15.5)

Example

Find the volume of the region in the first octant below the plane $2x + y - 2z = 2$ and $x \leqslant 1$, $y \leqslant 2$.

Solution: $V = \int_0^1 \Big[-(1-x)[2-2(1-x)] + \frac{1}{4}[4 - 4(1-x)^2] \Big]\, dx.$

$$V = \int_0^1 \big[-2(1-x) + 2(1-x)^2 + 1 - (1-x)^2 \big]\, dx,$$

$$V = \int_0^1 \big[-1 + 2x + (1-x)^2 \big]\, dx = \int_0^1 \big[-1 + 2x + 1 + x^2 - 2x \big]\, dx$$

$$V = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 \quad \Rightarrow \quad V = \frac{1}{3}. \qquad \triangleleft$$

Double integrals in polar coordinates. (Sect. 15.4)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

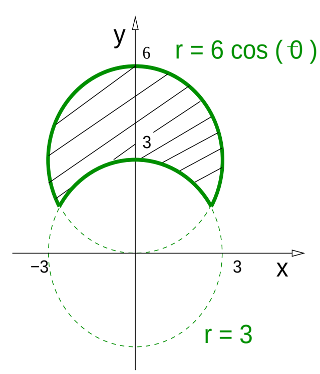
Solution: First sketch the integration region.

- $r = 6\sin(\theta)$ is a circle, since

$$r^2 = 6r\sin(\theta) \Leftrightarrow x^2 + y^2 = 6y$$

$$x^2 + (y-3)^2 = 3^2.$$

- The other curve is a circle $r = 3$ centered at the origin.



The condition $3 = r = 6\sin(\theta)$ determines the range in θ. Since $\sin(\theta) = 1/2$, we get $\theta_1 = 5\pi/6$ and $\theta_0 = \pi/6$.

Double integrals in polar coordinates. (Sect. 15.4)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

Solution: Recall: $\theta \in [\pi/6, 5\pi/6]$. The area is

$$A = \int_{\pi/6}^{5\pi/6} \int_{3}^{6\sin(\theta)} r dr\, d\theta = \int_{\pi/6}^{5\pi/6} \Big(\frac{r^2}{2}\Big|_3^{6\sin(\theta)} \Big)\, d\theta$$

$$A = \int_{\pi/6}^{5\pi/6} \Big[\frac{6^2}{2} \sin^2(\theta) - \frac{3^2}{2} \Big]\, d\theta = \int_{\pi/6}^{5\pi/6} \Big[\frac{6^2}{2^2} \big(1 - \cos(2\theta)\big) - \frac{3^2}{2} \Big]\, d\theta$$

$$A = 3^2 \Big(\frac{5\pi}{6} - \frac{\pi}{6} \Big) - \frac{3^2}{2} \Big(\sin(2\theta)\Big|_{\pi/6}^{5\pi/6} \Big) - \frac{3^2}{2} \Big(\frac{5\pi}{6} - \frac{\pi}{6} \Big).$$

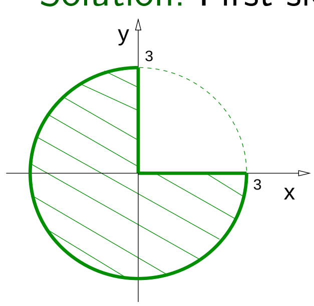
$$A = 6\pi - 3\pi - \frac{3^2}{2} \Big(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \Big),$$ hence $A = 3\pi + 9\sqrt{3}/2$. $\triangleleft$

Double integrals in polar coordinates. (Sect. 15.4)

Example

Use polar coordinate to compute the integral $\bar{y} = \dfrac{1}{A} \displaystyle\iint_R y\, dx\, dy$, where A is the area of the region in the plane given by the disk $x^2 + y^2 \leqslant 9$ minus the first quadrant.

Solution: First sketch the integration region.



Therefore, $A = \pi R^2 (3/4)$, with $R = 3$.
We use polar coordinates to compute $\bar{y}$.

$$\bar{y} = \frac{4}{27\pi} \int_{\pi/2}^{2\pi} \int_0^3 r\sin(\theta)\, r dr\, d\theta.$$

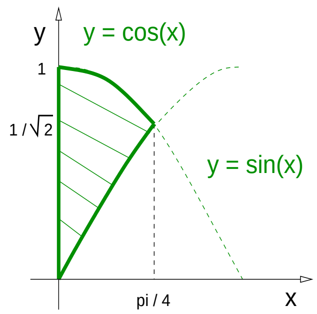
$$\bar{y} = \frac{4}{27\pi} \Big(-\cos(\theta)\Big|_{\pi/2}^{2\pi} \Big) \Big(\frac{r^3}{3}\Big|_0^3 \Big) = \frac{4}{27\pi} (-1)(9) \;\Rightarrow\; \bar{y} = -\frac{4}{3\pi}.$$ $\triangleleft$

Double integrals in Cartesian coordinates (Section 15.3)

Example

Switch the integration order in $I = \int_0^{\pi/4} \int_{\sin(x)}^{\cos(x)} dy\,dx$.

Solution: We first draw the integration region.



Divide the region at $y = \frac{1}{\sqrt{2}}$.

$$I = \int_0^{1/\sqrt{2}} \int_0^{\arcsin(y)} dx\,dy + \int_{1/\sqrt{2}}^{1} \int_0^{\arccos(y)} dx\,dy.$$

So, $I = \int_0^{1/\sqrt{2}} \int_0^{\arcsin(y)} dx\,dy + \int_{1/\sqrt{2}}^{1} \int_0^{\arccos(y)} dx\,dy.$ ◁

Double integrals in Cartesian coordinates (Section 15.2)

Example

Switch the integration order in $I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx$.

Solution:

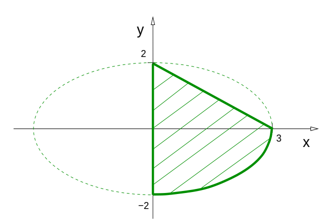
We first draw the integration region. Start with the outer limits.

$x \in [0, 3]$.

$y \leqslant 2 - 2x/3$ and $y \geqslant 2\sqrt{1 - \frac{x^2}{3^2}}$.

The lower limit is part of the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$$

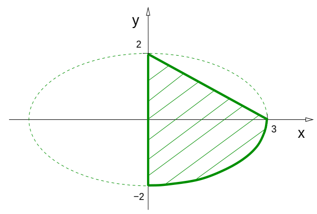


Double integrals in Cartesian coordinates (Section 15.2)

Example

Switch the integration order in $I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx.$

Solution:



Split the integral at $y = 0$.
In $y \in [-2, 0]$, holds $0 \leqslant x$.
The upper limit comes from
$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1,$
so, $x = +3\sqrt{1 - \frac{y^2}{2^2}}$.

In $y \in [0, 2]$, holds $0 \leqslant x$. The upper limit comes from $y = 2\Big(1 - \frac{x}{3}\Big)$, that is, $x = 3\Big(1 - \frac{y}{2}\Big)$. We then conclude:

$$I = \int_{-2}^{0} \int_0^{3\sqrt{1-\frac{y^2}{2^2}}} f(x,y)\,dx\,dy + \int_0^2 \int_0^{3(1-\frac{y}{2})} f(x,y)\,dx\,dy.$$

$\triangleleft$

Local and absolute extrema (Section 14.7)

Example

(a) Find all the critical points of $f(x,y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) $\nabla f(x,y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle$, then,

$$x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x-4) = 0.$$

There are two solutions, $x = 0 \Rightarrow y = 0$, and $x = 4 \Rightarrow y = 8$.

That is, there are two critical points, $(0,0)$ and $(4,8)$.

Local and absolute extrema (Section 14.7)

Example

(a) Find all the critical points of $f(x,y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:

(b) Recalling $\nabla f(x,y) = \langle 12y - 6x^2, 12x - 6y \rangle$, we compute

$$f_{xx} = -12x, \quad f_{yy} = -6, \quad f_{xy} = 12.$$

$$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 144 \left(\frac{x}{2} - 1 \right),$$

Since $D(0,0) = -144 < 0$, the point $(0,0)$ is a saddle point of f.

Since $D(4,8) = 144(2-1) > 0$, and $f_{xx}(4,8) = (-12)4 < 0$, the point $(4,8)$ is a local maximum of f. $\triangleleft$

Local and absolute extrema (Section 14.7)

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

We start finding the critical points inside the triangular region.

$$\nabla f(x,y) = \left\langle y - 2, x - \frac{1}{2}y \right\rangle = \langle 0, 0 \rangle, \quad \Rightarrow \quad y = 2, \quad y = 2x.$$

The solution is $(1,2)$. This point is outside in the triangular region given by the problem, so there is no critical point inside the region.

Local and absolute extrema (Section 14.7)

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

We now find the candidates for absolute maximum and minimum on the borders of the triangular region. We first record the boundary vertices:

$$\begin{aligned} (0,0) &\quad\Rightarrow\quad f(0,0) = 2, \\ (1,0) &\quad\Rightarrow\quad f(1,0) = 0, \\ (0,2) &\quad\Rightarrow\quad f(0,2) = 1. \end{aligned}$$

Local and absolute extrema (Section 14.7)

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

- The horizontal side of the triangle, $y = 0$, $x \in (0,1)$. Since

  $$g(x) = f(x,0) = 2 - 2x, \quad \Rightarrow \quad g'(x) = -2 \neq 0.$$

  there are no candidates in this part of the boundary.
- The vertical side of the triangle is $x = 0$, $y \in (0,2)$. Then,

  $$g(y) = f(0,y) = 2 - \frac{1}{4}y^2, \quad \Rightarrow \quad g'(y) = -\frac{1}{2}y = 0,$$

  so $y = 0$ and we recover the point $(0,0)$.

Local and absolute extrema (Section 14.7)

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

- The hypotenuse of the triangle $y = 2 - 2x$, $x \in (0,1)$. Then,

$$\begin{aligned} g(x) = f(x, 2-2x) &= 2 + x(2-2x) - 2x - \frac{1}{4}(2-2x)^2, \\ &= 2 + 2x - 2x^2 - 2x - (x^2 - 2x + 1), \\ &= 1 + 2x - 3x^2. \end{aligned}$$

  Then, $g'(x) = 2 - 6x = 0$ implies $x = \frac{1}{3}$, hence $y = \frac{4}{3}$. The candidate is $\left(\frac{1}{3}, \frac{4}{3}\right)$.

Local and absolute extrema (Section 14.7)

Example

Find the absolute maximum and absolute minimum of $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$.

Solution:

- Recall that we have obtained a candidate point $\left(\frac{1}{3}, \frac{4}{3}\right)$. We evaluate f at this point,

$$f\left(\frac{1}{3}, \frac{4}{3}\right) = 2 + \frac{4}{9} - \frac{2}{3} - \frac{1}{4}\frac{16}{9} = \frac{4}{3}.$$

Recalling that $f(0,0) = 2$, $f(1,0) = 0$, and $f(0,2) = 1$, the absolute maximum is at $(0,0)$, and the minimum is at $(1,0)$. ◁