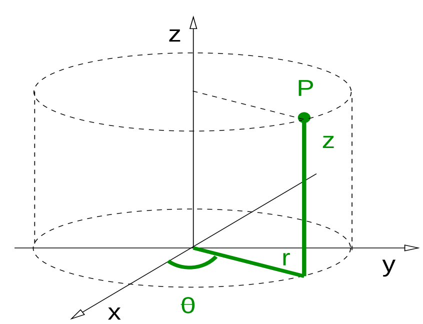
Integrals in cylindrical, spherical coordinates (Sect. 15.7)

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - Spherical coordinates in space.
  - Triple integral in spherical coordinates.

Cylindrical coordinates in space.

Definition
The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (r, θ, z) defined by the picture.



Remark: Cylindrical coordinates are just polar coordinates on the plane $z = 0$ together with the vertical coordinate z.

Theorem (Cartesian-cylindrical transformations)
The Cartesian coordinates of a point $P = (r, \theta, z)$ are given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.

The cylindrical coordinates of a point $P = (x, y, z)$ in the first and fourth quadrant are $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and $z = z$.

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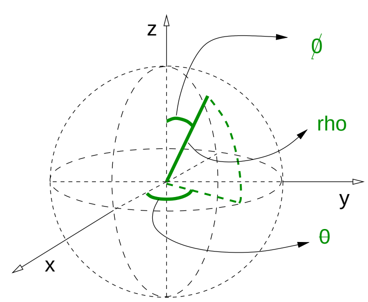
Spherical coordinates in $\mathbb{R}^3$

Definition

The *spherical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (ρ, ϕ, θ) defined by the picture.



Theorem (Cartesian-spherical transformations)

The Cartesian coordinates of $P = (\rho, \phi, \theta)$ *in the first quadrant are given by* $x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, *and* $z = \rho\cos(\phi)$.

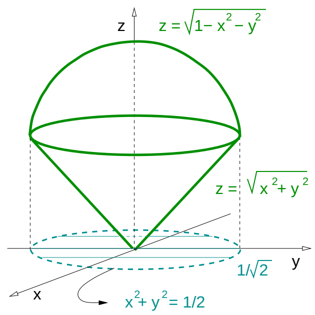
The spherical coordinates of $P = (x, y, z)$ *in the first quadrant are* $\rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arctan\Big(\dfrac{y}{x}\Big)$, *and* $\phi = \arctan\Big(\dfrac{\sqrt{x^2+y^2}}{z}\Big)$.

Spherical coordinates in $\mathbb{R}^3$

Example

Use spherical coordinates to express region between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: $(x = \rho \sin(\phi)\cos(\theta),\ y = \rho\sin(\phi)\sin(\theta),\ z = \rho\cos(\phi).)$



The top surface is the sphere $\rho = 1$.
The bottom surface is the cone:

$$\rho\cos(\phi) = \sqrt{\rho^2\sin^2(\phi)}$$

$$\cos(\phi) = \sin(\phi),$$

so the cone is $\phi = \dfrac{\pi}{4}$.

Hence: $R = \Big\{(\rho, \phi, \theta)\ :\ \theta \in [0, 2\pi],\ \phi \in \Big[0, \dfrac{\pi}{4}\Big],\ \rho \in [0, 1]\Big\}$.

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Triple integral in spherical coordinates

Theorem

If the function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in spherical coordinates as follows,

$$\iiint_R f\, dv = \iiint_R f(\rho, \phi, \theta)\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Remark:

- Spherical coordinates are useful when the integration region R is described in a simple way using spherical coordinates.
- Notice the extra factor $\rho^2 \sin(\phi)$ on the right-hand side.

Triple integral in spherical coordinates

Example

Find the volume of a sphere of radius R.

Solution: Sphere: $S = \{\theta \in [0, 2\pi],\ \phi \in [0, \pi],\ \rho \in [0, R]\}$.

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta,$$

$$V = \Big[\int_0^{2\pi} d\theta\Big] \Big[\int_0^{\pi} \sin(\phi)\, d\phi\Big] \Big[\int_0^{R} \rho^2\, d\rho\Big],$$

$$V = 2\pi \Big[-\cos(\phi)\Big|_0^{\pi}\Big] \frac{R^3}{3},$$

$$V = 2\pi \big[-\cos(\pi) + \cos(0)\big] \frac{R^3}{3};$$

hence: $V = \frac{4}{3}\pi R^3$. ◁

Triple integral in spherical coordinates

Example

Use spherical coordinates to find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.

Solution: $R = \Big\{(\rho, \phi, \theta) \ : \ \theta \in [0, 2\pi], \ \phi \in \Big[0, \frac{\pi}{4}\Big], \ \rho \in [0, 1]\Big\}$.

The calculation is simple, the region is a simple section of a sphere.

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta,$$

$$V = \Big[\int_0^{2\pi} d\theta\Big]\Big[\int_0^{\pi/4} \sin(\phi)\, d\phi\Big]\Big[\int_0^1 \rho^2\, d\rho\Big],$$

$$V = 2\pi \Big[-\cos(\phi)\Big|_0^{\pi/4}\Big]\Big(\frac{\rho^3}{3}\Big|_0^1\Big),$$

$$V = 2\pi \Big[-\frac{\sqrt{2}}{2} + 1\Big]\frac{1}{3} \quad \Rightarrow \quad V = \frac{\pi}{3}(2 - \sqrt{2}). \qquad \triangleleft$$

Triple integral in spherical coordinates

Example

Find the integral of $f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$ in the region $R = \{x \geqslant 0, \ y \geqslant 0, \ z \geqslant 0, \ x^2 + y^2 + z^2 \leqslant 1\}$ using spherical coordinates.

Solution: $R = \Big\{\theta \in \Big[0, \frac{\pi}{2}\Big], \ \phi \in \Big[0, \frac{\pi}{2}\Big], \ \rho \in [0, 1]\Big\}$. Hence,

$$I = \iiint_R f\, dv = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 e^{\rho^3} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta,$$

$$I = \Big[\int_0^{\pi/2} d\theta\Big]\Big[\int_0^{\pi/2} \sin(\phi)\, d\phi\Big]\Big[\int_0^1 e^{\rho^3} \rho^2\, d\rho\Big].$$

Use substitution: $u = \rho^3$, hence $du = 3\rho^2\, d\rho$, so

$$I = \frac{\pi}{2}\Big[-\cos(\phi)\Big|_0^{\frac{\pi}{2}}\Big] \int_0^1 \frac{e^u}{3}\, du \ \Rightarrow \ \iiint_R f\, dv = \frac{\pi}{6}(e - 1). \quad \triangleleft$$

Triple integral in spherical coordinates

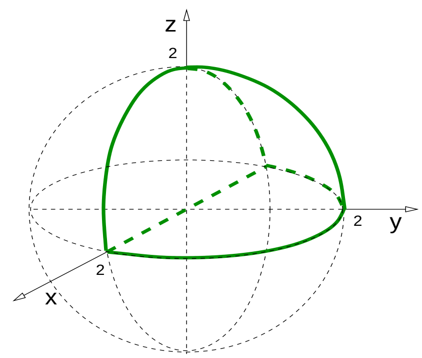
Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

Solution: ($x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.)

- Limits in x: $|x| \leqslant 2$;
- Limits in y: $0 \leqslant y \leqslant \sqrt{4-x^2}$, so the positive side of the disk $x^2 + y^2 \leqslant 4$.
- Limits in z: $0 \leqslant z \leqslant \sqrt{4-x^2-y^2}$, so a positive quarter of the ball $x^2+y^2+z^2 \leqslant 4$.



Triple integral in spherical coordinates

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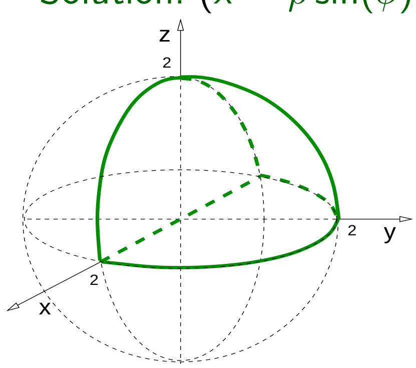

- Limits in θ: $\theta \in [0.\pi]$;
- Limits in ϕ: $\phi \in [0, \pi/2]$;
- Limits in ρ: $\rho \in [0, 2]$.
- The function to integrate is: $f = \rho^2 \sin(\phi)\sin(\theta)$.

$$I = \int_{0}^{\pi} \int_{0}^{\pi/2} \int_{0}^{2} \rho^2 \sin(\phi)\sin(\theta)\big(\rho^2\sin(\phi)\big)\, d\rho\, d\phi\, d\theta.$$

Triple integral in spherical coordinates

Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$$

Solution: $I = \int_0^{\pi}\int_0^{\pi/2}\int_0^{2} \rho^2\sin(\phi)\sin(\theta)\big(\rho^2\sin(\phi)\big)\,d\rho\,d\phi\,d\theta.$

$$I = \Big[\int_0^{\pi}\sin(\theta)\,d\theta\Big]\Big[\int_0^{\pi/2}\sin^2(\phi)\,d\phi\Big]\Big[\int_0^{2}\rho^4\,d\rho\Big],$$

$$I = \Big(-\cos(\theta)\Big|_0^{\pi}\Big)\Big[\int_0^{\pi/2}\frac{1}{2}\big(1-\cos(2\phi)\big)\,d\phi\Big]\Big(\frac{\rho^5}{5}\Big|_0^{2}\Big),$$

$$I = 2\,\frac{1}{2}\Big[\Big(\frac{\pi}{2}-0\Big)-\frac{1}{2}\Big(\sin(2\phi)\Big|_0^{\pi/2}\Big)\Big]\frac{2^5}{5} \quad\Rightarrow\quad I = \frac{2^4\pi}{5}. \quad \triangleleft$$

Triple integral in spherical coordinates

Example

Compute the integral $I = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec(\phi)}^{2} 3\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta.$

Solution: Recall: $\sec(\phi) = 1/\cos(\phi)$.

$$I = 2\pi\int_0^{\pi/3}\Big(\rho^3\Big|_{\sec(\phi)}^{2}\Big)\sin(\phi)\,d\phi,$$

$$I = 2\pi\int_0^{\pi/3}\Big(2^3-\frac{1}{\cos^3(\phi)}\Big)\sin(\phi)\,d\phi$$

In the second term substitute: $u=\cos(\phi)$, $du = -\sin(\phi)\,d\phi$.

$$I = 2\pi\Big[2^3\Big(-\cos(\phi)\Big|_0^{\pi/3}\Big)+\int_1^{1/2}\frac{du}{u^3}\Big].$$

Triple integral in spherical coordinates

Example

Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^{2} 3\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$

Solution: $I = 2\pi \Big[2^3 \Big(-\cos(\phi)\Big|_0^{\pi/3} \Big) + \int_1^{1/2} \frac{du}{u^3} \Big].$

$$I = 2\pi \Big[2^3 \Big(-\frac{1}{2} + 1 \Big) - \int_{1/2}^{1} u^{-3} du \Big] = 2\pi \Big[4 - \Big(\frac{u^{-2}}{-2} \Big|_{1/2}^{1} \Big) \Big],$$

$$I = 2\pi \Big[4 + \frac{1}{2} \big(u^{-2} \Big|_{1/2}^{1} \big) \Big] = 2\pi \Big[4 + \frac{1}{2} (1 - 2^2) \Big] = 2\pi \Big[\frac{8}{2} - \frac{3}{2} \Big]$$

We conclude: $I = 5\pi$. ◁