# Integrals in cylindrical, spherical coordinates (Sect. 15.7)

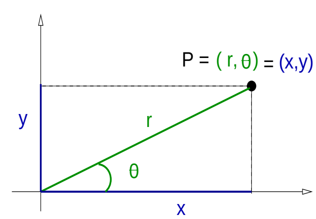
- Integration in cylindrical coordinates.
  - Review: Polar coordinates in a plane.
  - Cylindrical coordinates in space.
  - Triple integral in cylindrical coordinates.

Next class:

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - Spherical coordinates in space.
  - Triple integral in spherical coordinates.

# Review: Polar coordinates in plane

**Definition**
The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair $(r, \theta)$ defined by the picture.




**Theorem (Cartesian-polar transformations)**
*The Cartesian coordinates of a point $P = (r, \theta)$ are given by*

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

*The polar coordinates of a point $P = (x, y)$ in the first and fourth quadrant are*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

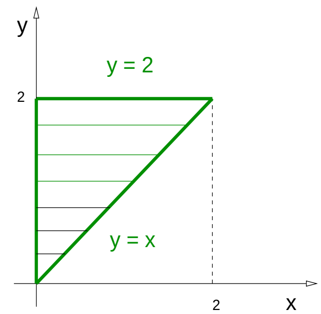
## Recall: Polar coordinates in a plane

### Example

Express in polar coordinates the integral $I = \int_0^2 \int_0^y x \, dx \, dy$.

Solution: Recall: $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

More often than not helps to sketch the integration region.




The outer integration limit: $y \in [0, 2]$.

Then, for every $y \in [0, 2]$ the $x$ coordinate satisfies $x \in [0, y]$.

The upper limit for $x$ is the curve $x = y$.

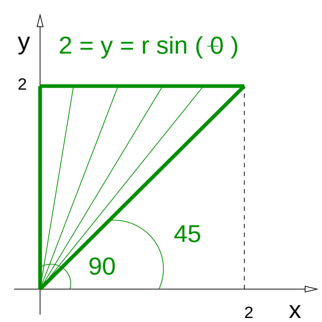
Now is simple to describe this domain in polar coordinates: The line $y = x$ is $\theta_0 = \pi/4$; the line $x = 0$ is $\theta_1 = \pi/2$.

## Recall: Polar coordinates in a plane

### Example

Express in polar coordinates the integral $I = \int_0^2 \int_0^y x \, dx \, dy$.

Solution: Recall: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $\theta_0 = \pi/4$, $\theta_1 = \pi/2$.




The lower integration limit in $r$ is $r = 0$.

The upper integration limit is $y = 2$, that is, $2 = y = r\sin(\theta)$.

Hence $r = 2/\sin(\theta)$.

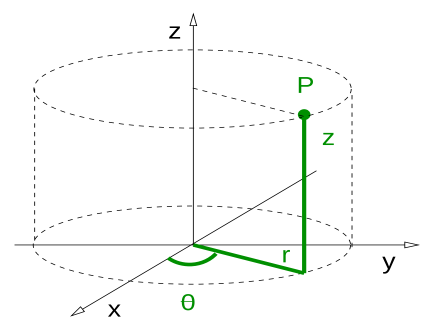
We conclude: $\int_0^2 \int_0^y x \, dx \, dy = \int_{\pi/4}^{\pi/2} \int_0^{2/\sin(\theta)} r\cos(\theta)(r\,dr)\,d\theta. \lhd$

# Integrals in cylindrical, spherical coordinates (Sect. 15.7)

- Integration in cylindrical coordinates.
  - Review: Polar coordinates in a plane.
  - **Cylindrical coordinates in space.**
  - Triple integral in cylindrical coordinates.

# Cylindrical coordinates in space

**Definition**
The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple $(r, \theta, z)$ defined by the picture.




**Remark:** Cylindrical coordinates are just polar coordinates on the plane $z = 0$ together with the vertical coordinate $z$.

**Theorem (Cartesian-cylindrical transformations)**
*The Cartesian coordinates of a point $P = (r, \theta, z)$ in the first quadrant are given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.*

*The cylindrical coordinates of a point $P = (x, y, z)$ in the first quadrant are given by $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and $z = z$.*

## Cylindrical coordinates in space

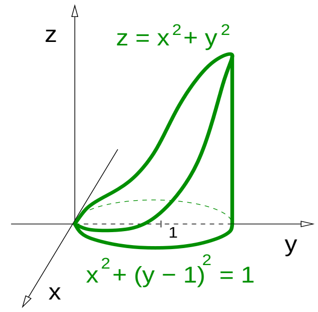
### Example

Use cylindrical coordinates to describe the region
$R = \{(x, y, z) \;:\; x^2 + (y-1)^2 \leqslant 1,\; 0 \leqslant z \leqslant x^2 + y^2\}$.

Solution: We first sketch the region.

The base of the region is at $z = 0$, given by the disk $x^2 + (y-1)^2 \leqslant 1$.

The top of the region is the paraboloid $z = x^2 + y^2$.




In cylindrical coordinates: $z = x^2 + y^2 \Leftrightarrow z = r^2$, and

$$x^2 + y^2 - 2y + 1 \leqslant 1 \;\Leftrightarrow\; r^2 - 2r\sin(\theta) \leqslant 0 \;\Leftrightarrow\; r \leqslant 2\sin(\theta)$$

Hence: $R = \{(r, \theta, z) \;:\; \theta \in [0, \pi],\; r \in [0, 2\sin(\theta)],\; z \in [0, r^2]\}.\triangleleft$

## Integrals in cylindrical, spherical coordinates (Sect. 15.7)

- Integration in cylindrical coordinates.
  - Review: Polar coordinates in a plane.
  - Cylindrical coordinates in space.
  - **Triple integral in cylindrical coordinates.**

## Triple integrals using cylindrical coordinates

### Theorem

*If the function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then the triple integral of function $f$ in the region $R$ can be expressed in cylindrical coordinates as follows,*

$$\iiint_R f\, dv = \iiint_R f(r, \theta, z)\, r\, dr\, d\theta\, dz.$$

### Remark:

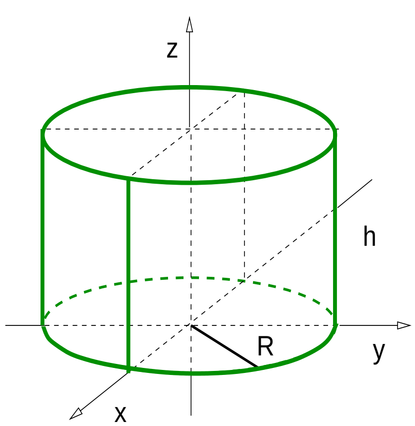
- Cylindrical coordinates are useful when the integration region $R$ is described in a simple way using cylindrical coordinates.
- Notice the extra factor $r$ on the right-hand side.

## Triple integrals using cylindrical coordinates

### Example

Find the volume of a cylinder of radius $R$ and height $h$.

Solution: $R = \{(r, \theta, z) \ :\ \theta \in [0, 2\pi],\ r \in [0, R],\ z \in [0, h]\}$.




$$V = \int_0^{2\pi} \int_0^R \int_0^h dz\, (r\, dr)\, d\theta,$$

$$V = h \int_0^{2\pi} \int_0^R r\, dr\, d\theta,$$

$$V = h \frac{R^2}{2} \int_0^{2\pi} d\theta,$$

$$V = h \frac{R^2}{2} 2\pi,$$

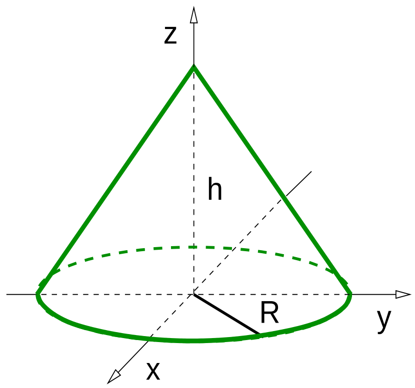
We conclude: $V = \pi R^2 h$. $\triangleleft$

## Triple integrals using cylindrical coordinates

**Example**

Find the volume of a cone of base radius $R$ and height $h$.

**Solution:** $R = \Big\{\theta \in [0, 2\pi],\ r \in [0, R],\ z \in \Big[0, -\frac{h}{R} r + h\Big]\Big\}$.




$$V = \int_0^{2\pi} \int_0^R \int_0^{h(1-r/R)} dz\,(r\,dr)\,d\theta,$$

$$V = h \int_0^{2\pi} \int_0^R \Big(1 - \frac{r}{R}\Big) r\,dr\,d\theta,$$

$$V = h \int_0^{2\pi} \int_0^R \Big(r - \frac{r^2}{R}\Big)\,dr\,d\theta,$$

$$V = h\Big(\frac{R^2}{2} - \frac{R^3}{3R}\Big) \int_0^{2\pi} d\theta = 2\pi h R^2 \frac{1}{6}.$$

We conclude: $V = \frac{1}{3}\pi R^2 h$. $\triangleleft$

## Triple integrals using cylindrical coordinates

**Example**

Sketch the region with volume $V = \displaystyle\int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{9-r^2}} r dz\,dr\,d\theta$.

**Solution:** The integration region is

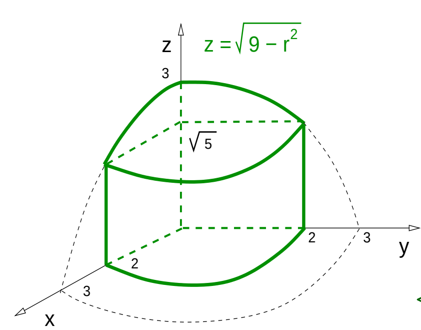
$$R = \{(r, \theta, z)\ :\ \theta \in [0, \pi/2],\ r \in [0, 2],\ z \in [0, \sqrt{9 - r^2}]\}.$$

We upper boundary is a sphere,

$z^2 = 9 - r^2 \Leftrightarrow x^2 + y^2 + z^2 = 3^2.$

The upper limit for $r$ is $r = 2$, so

$$z = \sqrt{9 - 2^2} \Rightarrow z = \sqrt{5}.$$




$\triangleleft$

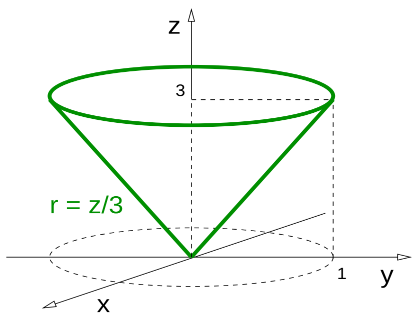
## Triple integrals using cylindrical coordinates

### Example

Change the integration order and compute the integral

$$I = \int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 dr\, dz\, d\theta.$$

Solution: First, sketch the integration region. Start from the outer limits to the inner limits.




$$I = \int_0^{2\pi} \int_0^1 \int_{3r}^3 dz\, r^3 dr\, d\theta$$

$$V = 2\pi \int_0^1 \left( z \Big|_{3r}^3 \right) r^3 dr$$

$$V = 2\pi \int_0^1 3(r^3 - r^4)\, dr$$

$$V = 6\pi \left( \frac{r^4}{4} - \frac{r^5}{5} \right) \Big|_0^1.$$

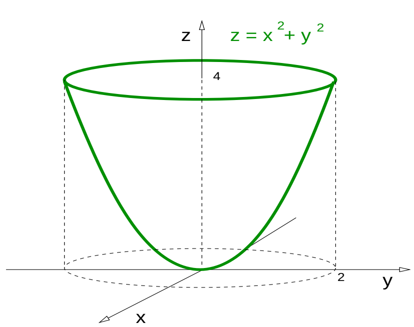
So, $I = 6\pi \frac{1}{20}$, that is, $I = \frac{3\pi}{10}$. ◁

## Triple integrals using cylindrical coordinates

### Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region in space $R = \{(x, y, z) \;:\; x^2 + y^2 \leqslant 2^2,\; x^2 + y^2 \leqslant z \leqslant 4\}$.

Solution:




The symmetry of the region implies $\bar{x} = 0$ and $\bar{y} = 0$. (We verify this result later on.) We only need to compute $\bar{z}$.

Since $\bar{z} = \frac{1}{V} \iiint_R z\, dv$, we start computing the total volume $V$.

We use cylindrical coordinates.

$$V = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 dz\, r dr\, d\theta = 2\pi \int_0^2 \left( z \Big|_{r^2}^4 \right) r dr = 2\pi \int_0^2 (4r - r^3) dr.$$

## Triple integrals using cylindrical coordinates

Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \overline{x}, \overline{y}, \overline{z} \rangle$ of the region in space $R = \{(x, y, z) \ : \ x^2 + y^2 \leqslant 2^2, \ x^2 + y^2 \leqslant z \leqslant 4\}$.

Solution: $V = 2\pi \int_0^2 (4r - r^3)\, dr = 2\pi \Big[ 4\Big(\frac{r^2}{2}\Big|_0^2\Big) - \Big(\frac{r^4}{4}\Big|_0^2\Big)\Big]$.

Hence $V = 2\pi(8 - 4)$, so $V = 8\pi$. Then, $\overline{z}$ is given by,

$$\overline{z} = \frac{1}{8\pi} \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z dz \, r dr \, d\theta = \frac{2\pi}{8\pi} \int_0^2 \Big(\frac{z^2}{2}\Big|_{r^2}^4\Big)\, r dr;$$

$$\overline{z} = \frac{1}{8} \int_0^2 (16r - r^5)\, dr = \frac{1}{8}\Big[16\Big(\frac{r^2}{2}\Big|_0^2\Big) - \Big(\frac{r^6}{6}\Big|_0^2\Big)\Big];$$

$$\overline{z} = \frac{1}{8}\Big(32 - \frac{64}{6}\Big) = 4 - \frac{4}{3} \quad \Rightarrow \quad \overline{z} = \frac{8}{3}.$$

## Triple integrals using cylindrical coordinates

Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \overline{x}, \overline{y}, \overline{z} \rangle$ of the region in space $R = \{(x, y, z) \ : \ x^2 + y^2 \leqslant 2^2, \ x^2 + y^2 \leqslant z \leqslant 4\}$.

Solution: We obtained $\overline{z} = \frac{8}{3}$.

It is simple to see that $\overline{x} = 0$ and $\overline{y} = 0$. For example,

$$\overline{x} = \frac{1}{8\pi} \int_0^{2\pi} \int_0^2 \int_{r^2}^4 \big[r \cos(\theta)\big]\, dz \, r dr \, d\theta$$
$$= \frac{1}{8\pi} \Big[\int_0^{2\pi} \cos(\theta) d\theta\Big] \Big[\int_0^2 \int_{r^2}^4 dz \, r^2 dr\Big].$$

But $\int_0^{2\pi} \cos(\theta) d\theta = \sin(2\pi) - \sin(0) = 0$, so $\overline{x} = 0$.

A similar calculation shows $\overline{y} = 0$. Hence $\bar{\mathbf{r}} = \langle 0, 0, 8/3 \rangle$. $\triangleleft$