Triple integrals in Cartesian coordinates (Sect. 15.5)

- Review: Triple integrals in arbitrary domains.
- Examples: Changing the order of integration.
- The average value of a function in a region in space.
- Triple integrals in arbitrary domains.

Review: Triple integrals in arbitrary domains

Theorem

If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the domain

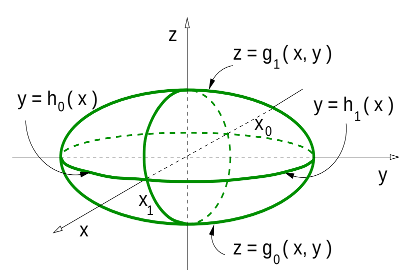
$$D = \big\{x \in [x_0, x_1],\ y \in [h_0(x), h_1(x)],\ z \in [g_0(x,y), g_1(x,y)]\big\},$$

where $g_0, g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ are continuous, then the triple integral of the function f in the region D is given by

$$\iiint_D f\, dv = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

Example

In the case that D is an ellipsoid, the figure represents the graph of functions g_1, g_0 and h_1, h_0.



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Changing the order of integration

Example

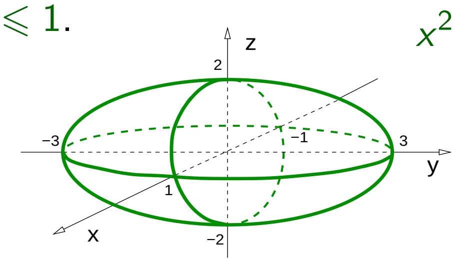
Change the order of integration in the triple integral

$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx.$$

Solution: First: Sketch the integration region.

Start from the outer integration limits to the inner limits.

- Limits in x: $x \in [-1, 1]$.
- Limits in y: $|y| \leqslant 3\sqrt{1-x^2}$, so, $x^2 + \dfrac{y^2}{3^2} \leqslant 1$.
- The limits in z: $|z| \leqslant 2\sqrt{1 - x^2 - \dfrac{y^2}{3^2}}$, so, $x^2 + \dfrac{y^2}{3^2} + \dfrac{z^2}{2^2} \leqslant 1$.



Changing the order of integration

Example

Change the order of integration in the triple integral

$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx.$$

Solution: Region: $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} \leqslant 1$. We conclude:

$$V = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy\, dz\, dx.$$

$$V = \int_{-2}^{2} \int_{-\sqrt{1-(z/2)^2}}^{\sqrt{1-(z/2)^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy\, dx\, dz.$$

$$V = \int_{-2}^{2} \int_{-3\sqrt{1-(z/2)^2}}^{3\sqrt{1-(z/2)^2}} \int_{-\sqrt{1-(y/3)^2-(z/2)^2}}^{\sqrt{1-(y/3)^2-(z/2)^2}} dx\, dy\, dz. \qquad \triangleleft$$

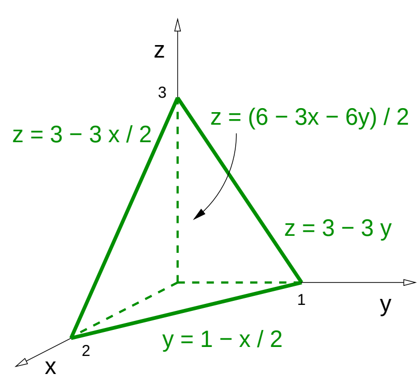
Changing the order of integration

Example

Interchange the limits in $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

Solution: Recall: *Sketch the integration region starting from the outer integration limits to the inner integration limits.*

- $x \in [0, 2]$.
- $y \in \left[0, 1 - \frac{x}{2}\right]$ so the upper limit is the line $y = 1 - \frac{x}{2}$.
- $z \in \left[0, 3 - \frac{3x}{2} - 3y\right]$ so the upper limit is the plane $z = 3 - \frac{3x}{2} - 3y$. This plane contains the points $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$.

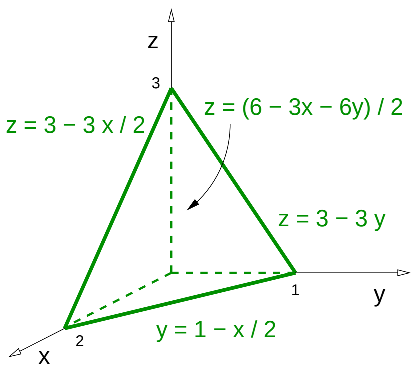


Changing the order of integration

Example

Interchange the limits in $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

Solution: The region: $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $6 \geqslant 3x + 6y + 2z$.



$$V = \int_0^3 \int_0^{1-z/3} \int_0^{2-2y-2z/3} dx\, dy\, dz.$$

$$V = \int_0^1 \int_0^{3-3y} \int_0^{2-2y-2z/3} dx\, dz\, dy.$$

$$V = \int_0^2 \int_0^{3-3x/2} \int_0^{1-x/2-z/3} dy\, dz\, dx.$$

$$V = \int_0^3 \int_0^{2-2z/3} \int_0^{1-x/2-z/3} dy\, dx\, dz.$$

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- **The average value of a function in a region in space.**
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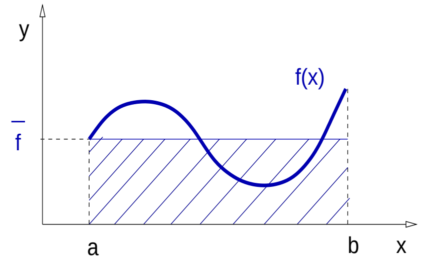
Average value of a function in a region in space

Definition (Review: 1-variable)

The *average* of a function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{(b-a)} \int_a^b f(x)\, dx.$$



Definition

The *average* of a function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ on the region R with volume V, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{V} \iiint_R f\, dv.$$

Average value of a function in a region in space

Example

Find the average of $f(x, y, z) = xyz$ in the first octant bounded by the planes $x = 1$, $y = 2$, $z = 3$.

Solution: The volume of the rectangular integration region is

$$V = \int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx \quad \Rightarrow \quad V = 6.$$

The average of function f is:

$$\overline{f} = \frac{1}{6} \int_0^1 \int_0^2 \int_0^3 xyz\, dz\, dy\, dx = \frac{1}{6} \Big[\int_0^1 x\, dx\Big] \Big[\int_0^2 y\, dy\Big] \Big[\int_0^3 z\, dz\Big]$$

$$\overline{f} = \frac{1}{6} \Big(\frac{x^2}{2}\Big|_0^1\Big) \Big(\frac{y^2}{2}\Big|_0^2\Big) \Big(\frac{z^2}{2}\Big|_0^3\Big) = \frac{1}{6} \Big(\frac{1}{2}\Big) \Big(\frac{4}{2}\Big) \Big(\frac{9}{2}\Big) \Rightarrow \overline{f} = 1/4.$$

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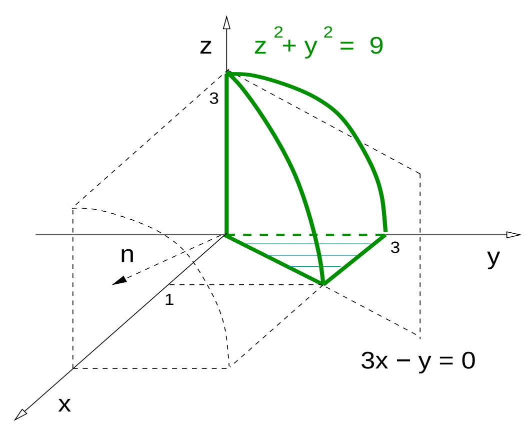
Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall: *Sketch the integration region.*

- The integration region is in the first octant.
- It is inside the cylinder $y^2 + z^2 = 9$.
- It is on one side of the plane $3x - y = 0$. The plane has normal vector $\mathbf{n} = \langle 3, -1, 0 \rangle$ and contains $(0, 0, 0)$.

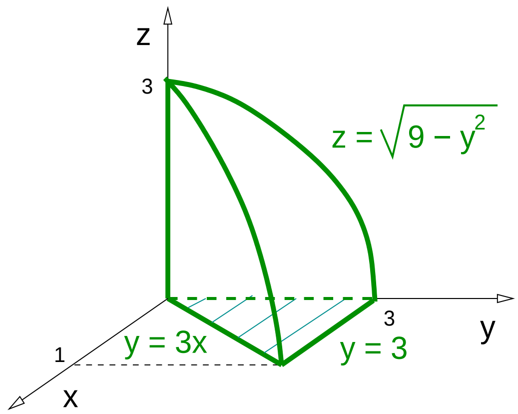


Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: We have found the region:



The integration limits are:

- Limits in z: $0 \leqslant z \leqslant \sqrt{9 - y^2}$.
- Limits in x: $0 \leqslant x \leqslant y/3$.
- Limits in y: $0 \leqslant y \leqslant 3$.

We obtain $I = \int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z\, dz\, dx\, dy$.

Triple integrals in arbitrary domains

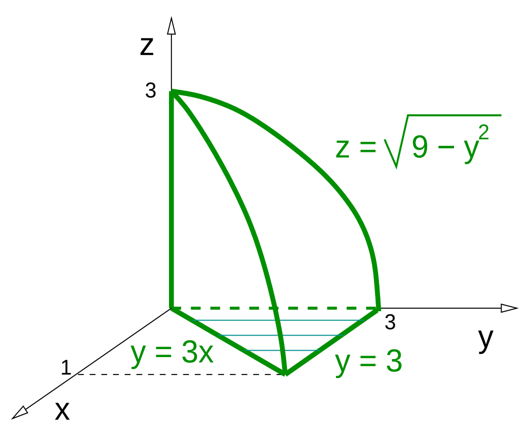
Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall:

$$\int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z\, dz\, dx\, dy.$$

Just for practice, let us change the integration order to $dz\, dy\, dx$:



The result is: $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx$.

Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall: $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx$.

We now compute the integral:

$$I = \int_0^1 \int_{3x}^3 \Big(\frac{z^2}{2} \Big|_0^{\sqrt{9-y^2}} \Big) \, dy \, dx,$$

$$I = \frac{1}{2} \int_0^1 \int_{3x}^3 (9 - y^2) dy \, dx,$$

$$I = \frac{1}{2} \int_0^1 \Big[9 \Big(y \Big|_{3x}^3 \Big) - \Big(\frac{y^3}{3} \Big|_{3x}^3 \Big) \Big] \, dx.$$

Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall: $I = \frac{1}{2} \int_0^1 \Big[9 \Big(y \Big|_{3x}^3 \Big) - \Big(\frac{y^3}{3} \Big|_{3x}^3 \Big) \Big] dx$.

Therefore,

$$I = \frac{1}{2} \int_0^1 \Big[27(1 - x) - 9(1 - x)^3 \Big] dx,$$

$$I = \frac{9}{2} \int_0^1 \Big[3(1 - x) - (1 - x)^3 \Big] dx.$$

Substitute $u = 1 - x$, then $du = -dx$, so, $I = \frac{9}{2} \int_0^1 (3u - u^3) du$.

Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall: $I = \frac{9}{2} \int_0^1 (3u - u^3)\,du$.

$$I = \frac{9}{2} \int_0^1 (3u - u^3)\,du,$$

$$I = \frac{9}{2}\Big[3\Big(\frac{u^2}{2}\Big|_0^1\Big) - \Big(\frac{u^4}{4}\Big|_0^1\Big)\Big] = \frac{9}{2}\Big(\frac{3}{2} - \frac{1}{4}\Big).$$

We conclude $\iiint_D f\,dv = \frac{45}{8}$. $\triangleleft$