Review: Triple integrals in arbitrary domains.

Theorem

If \( f : D \subset \mathbb{R}^3 \rightarrow \mathbb{R} \) is continuous in the domain

\[
D = \{ x \in [x_0, x_1], y \in [h_0(x), h_1(x)], z \in [g_0(x,y), g_1(x,y)] \},
\]

where \( g_0, g_1 : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( h_0, h_1 : \mathbb{R} \rightarrow \mathbb{R} \) are continuous, then the triple integral of the function \( f \) in the region \( D \) is given by

\[
\int\int\int_D f \, dv = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x, y, z) \, dz \, dy \, dx.
\]

Example

In the case that \( D \) is an ellipsoid, the figure represents the graph of functions \( g_1, g_0 \) and \( h_1, h_0 \).
Triple integrals in Cartesian coordinates (Sect. 15.5)

- Review: Triple integrals in arbitrary domains.
- **Examples: Changing the order of integration.**
- The average value of a function in a region in space.
- Triple integrals in arbitrary domains.

Changing the order of integration

**Example**

Change the order of integration in the triple integral

\[
V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz \ dy \ dx.
\]

**Solution:** First: Sketch the integration region.

*Start from the outer integration limits to the inner limits.*

- Limits in \(x\): \(x \in [-1, 1]\).
- Limits in \(y\): \(|y| \leq 3\sqrt{1-x^2}\), so, \(x^2 + \frac{y^2}{3^2} \leq 1\).
- The limits in \(z\): \(|z| \leq 2\sqrt{1-x^2 - \frac{y^2}{3^2}}\), so, \(x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} \leq 1\).
Changing the order of integration

Example
Change the order of integration in the triple integral
\[ V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz \, dy \, dx. \]

Solution: Region: \( x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} \leq 1 \). We conclude:

\[ V = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy \, dz \, dx. \]
\[ V = \int_{-2}^{2} \int_{-\sqrt{1-(z/2)^2}}^{\sqrt{1-(z/2)^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy \, dx \, dz. \]
\[ V = \int_{-2}^{2} \int_{-3\sqrt{1-(z/2)^2}}^{3\sqrt{1-(z/2)^2}} \int_{-\sqrt{1-(y/3)^2-(z/2)^2}}^{\sqrt{1-(y/3)^2-(z/2)^2}} dx \, dy \, dz. \]
\[ \triangleq \]

Changing the order of integration

Example
Interchange the limits in \( V = \int_{0}^{2} \int_{0}^{1-x/2} \int_{0}^{3-3y-3x/2} dz \, dy \, dx. \)

Solution: Recall: Sketch the integration region starting from the outer integration limits to the inner integration limits.

\( x \in [0, 2]. \)
\( y \in \left[ 0, 1 - \frac{x}{2} \right] \) so the upper limit is the line \( y = 1 - \frac{x}{2}. \)
\( z \in \left[ 0, 3 - \frac{3x}{2} - 3y \right] \) so the upper limit is the plane \( z = 3 - \frac{3x}{2} - 3y. \)

This plane contains the points \( (2, 0, 0), (0, 1, 0) \) and \( (0, 0, 3). \)
Changing the order of integration

Example
Interchange the limits in $V = \int_{0}^{2} \int_{0}^{1-x/2} \int_{0}^{3-3y-3x/2} dz \, dy \, dx$.

Solution: The region: $x \geq 0$, $y \geq 0$, $z \geq 0$ and $6 \geq 3x + 6y + 2z$.

$V = \int_{0}^{3} \int_{0}^{1-z/3} \int_{0}^{2-2y-2z/3} dx \, dy \, dz$.

$V = \int_{0}^{1} \int_{0}^{3-3y} \int_{0}^{2-2y-2z/3} dx \, dz \, dy$.

$V = \int_{0}^{2} \int_{0}^{3-3x/2} \int_{0}^{1-x/2-z/3} dy \, dz \, dx$.

$V = \int_{0}^{3} \int_{0}^{2-2z/3} \int_{0}^{1-x/2-z/3} dy \, dx \, dz$.

Triple integrals in Cartesian coordinates (Sect. 15.5)

- Review: Triple integrals in arbitrary domains.
- Examples: Changing the order of integration.
- The average value of a function in a region in space.
- Triple integrals in arbitrary domains.
Average value of a function in a region in space

**Definition (Review: 1-variable)**

The average of a function $f : [a, b] \rightarrow \mathbb{R}$ on the interval $[a, b]$, denoted by $\bar{f}$, is given by

$$
\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
$$

**Definition**

The average of a function $f : R \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ on the region $R$ with volume $V$, denoted by $\bar{f}$, is given by

$$
\bar{f} = \frac{1}{V} \iiint_R f \, dv.
$$

---

**Example**

Find the average of $f(x, y, z) = xyz$ in the first octant bounded by the planes $x = 1$, $y = 2$, $z = 3$.

**Solution:** The volume of the rectangular integration region is

$$
V = \int_0^1 \int_0^2 \int_0^3 dz \, dy \, dx \quad \Rightarrow \quad V = 6.
$$

The average of function $f$ is:

$$
\bar{f} = \frac{1}{6} \int_0^1 \int_0^2 \int_0^3 xyz \, dz \, dy \, dx = \frac{1}{6} \left[ \int_0^1 x \, dx \right] \left[ \int_0^2 y \, dy \right] \left[ \int_0^3 z \, dz \right]
$$

$$
\bar{f} = \frac{1}{6} \left( \frac{x^2}{2} \right) \left( \frac{y^2}{2} \right) \left( \frac{z^3}{3} \right) = \frac{1}{6} \left( \frac{1}{2} \right) \left( \frac{4}{2} \right) \left( \frac{9}{2} \right) \quad \Rightarrow \quad \bar{f} = 1/4.
$$

$\triangle$
Triple integrals in Cartesian coordinates (Sect. 15.5)

- Review: Triple integrals in arbitrary domains.
- Examples: Changing the order of integration.
- The average value of a function in a region in space.
- **Triple integrals in arbitrary domains.**

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**Example**

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$.

**Solution:** Recall: *Sketch the integration region.*

- The integration region is in the first octant.
- It is inside the cylinder $y^2 + z^2 = 9$.
- It is on one side of the plane $3x - y = 0$. The plane has normal vector $n = \langle 3, -1, 0 \rangle$ and contains $(0, 0, 0)$. 

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![Diagram](image.png)
Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$.

Solution: We have found the region:

The integration limits are:

- Limits in $z$: $0 \leq z \leq \sqrt{9 - y^2}$.
- Limits in $x$: $0 \leq x \leq y/3$.
- Limits in $y$: $0 \leq y \leq 3$.

We obtain $I = \int_0^3 \int_0^{y/3} \int_0^{\sqrt{9 - y^2}} z \, dz \, dx \, dy$.

Solution: Recall:

$\int_0^3 \int_0^{y/3} \int_0^{\sqrt{9 - y^2}} z \, dz \, dx \, dy$.

Just for practice, let us change the integration order to $dz \, dy \, dx$:

The result is: $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9 - y^2}} z \, dz \, dy \, dx$. 
Example
Compute the triple integral of \( f(x, y, z) = z \) in the region bounded by \( x \geq 0, \ z \geq 0, \ y \geq 3x, \) and \( 9 \geq y^2 + z^2. \)

Solution: Recall: \[
I = \int_0^1 \int_3^{3x} \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx.
\]
We now compute the integral:
\[
I = \int_0^1 \int_3^{3x} \left( \frac{z^2}{2} \right) \, dy \, dx,
\]
\[
I = \frac{1}{2} \int_0^1 \int_3^{3x} (9 - y^2) \, dy \, dx,
\]
\[
I = \frac{1}{2} \int_0^1 \left[ 9 \left( y \bigg|_3^{3x} \right) - \left( \frac{y^3}{3} \bigg|_3^{3x} \right) \right] \, dx.
\]
Substitute \( u = 1 - x, \) then \( du = -dx, \) so, \[
I = \frac{9}{2} \int_0^1 (3u - u^3) \, du.
\]
Example
Compute the triple integral of \( f(x, y, z) = z \) in the region bounded by \( x \geq 0, z \geq 0, y \geq 3x, \) and \( 9 \geq y^2 + z^2. \)

Solution: Recall: \( I = \frac{9}{2} \int_0^1 (3u - u^3) du. \)

\[
I = \frac{9}{2} \int_0^1 (3u - u^3) du,
\]

\[
I = \frac{9}{2} \left[ 3 \left( \frac{u^2}{2} \bigg|_0^1 \right) - \left( \frac{u^4}{4} \bigg|_0^1 \right) \right] = \frac{9}{2} \left( \frac{3}{2} - \frac{1}{4} \right).
\]

We conclude \( \iiint_D f \, dv = \frac{45}{8}. \) \( \triangle \)