Triple integrals in Cartesian coordinates (Sect. 15.5)

- Triple integrals in rectangular boxes.
- Triple integrals in arbitrary domains.
- Volume on a region in space.

Triple integrals in rectangular boxes

Definition

The *triple integral* of a function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ in the rectangular box $R = [\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1] \times [\hat{z}_0, \hat{z}_1]$ is the number

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\, \Delta x \Delta y \Delta z$$

where $x_i^* \in [x_i, x_{i+1}]$, $y_j^* \in [y_j, y_{j+1}]$, $z_k^* \in [z_k, z_{k+1}]$ are sample points, while $\{x_i\}$, $\{y_j\}$, $\{z_k\}$, with $i, j, k = 0, \cdots, n$, are partitions of the intervals $[\hat{x}_0, \hat{x}_1]$, $[\hat{y}_0, \hat{y}_1]$, $[\hat{z}_0, \hat{z}_1]$, respectively, and

$$\Delta x = \frac{(\hat{x}_1 - \hat{x}_0)}{n}, \quad \Delta y = \frac{(\hat{y}_1 - \hat{y}_0)}{n}, \quad \Delta z = \frac{(\hat{z}_1 - \hat{z}_0)}{n}.$$

Triple integrals in rectangular boxes

Remark:

- A finite sum S_n below is called a Riemann sum, where

$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\, \Delta x \Delta y \Delta z.$$

- Then holds $\displaystyle\iiint_R f(x,y,z)\, dx\, dy\, dz = \lim_{n\to\infty} S_n.$

Theorem (Fubini)

If function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the rectangle $R = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, then holds

$$\iiint_R f(x,y,z)\, dx\, dy\, dz = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x,y,z)\, dz\, dy\, dx.$$

Furthermore, the integral above can be computed integrating the variables x, y, z in any order.

Triple integrals in rectangular boxes

Review: The Riemann sums and their limits.

Single variable functions in $[\hat{x}_0, \hat{x}_1]$:

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x = \int_{\hat{x}_0}^{\hat{x}_1} f(x) dx.$$

Two variable functions in $[\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1]$: (Fubini)

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\Delta x \Delta y = \int_{\hat{x}_0}^{\hat{x}_1} \int_{\hat{y}_0}^{\hat{y}_1} f(x,y)\, dy\, dx.$$

Three variable functions in $[\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1] \times [\hat{z}_0, \hat{z}_1]$: (Fubini)

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\Delta x \Delta y \Delta z = \int_{\hat{x}_0}^{\hat{x}_1} \int_{\hat{y}_0}^{\hat{y}_1} \int_{\hat{z}_0}^{\hat{z}_1} f(x,y,z) dz\, dy\, dx$$

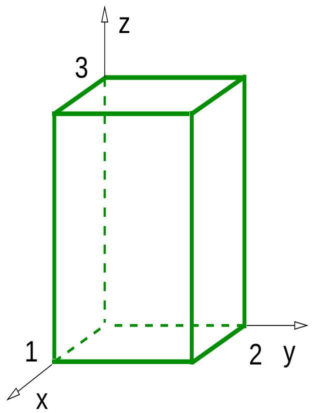
Triple integrals in rectangular boxes

Example

Compute the integral of $f(x,y,z) = xyz^2$ on the domain $R = [0,1] \times [0,2] \times [0,3]$.

Solution: It is useful to sketch the integration region first:

$$R = \{(x,y,z) \in \mathbb{R}^3 : \; x \in [0,1], \; y \in [0,2], \; z \in [0,3]\}.$$



The integral we need to compute is

$$\iiint_R f \, dv = \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx,$$

where we denoted $dv = dx\,dy\,dz$.

Triple integrals in rectangular boxes

Example

Compute the integral of $f(x,y,z) = xyz^2$ on the domain $R = [0,1] \times [0,2] \times [0,3]$.

Solution: $\displaystyle \iiint_R f \, dV = \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx.$

We have chosen a particular integration order. (Recall: Since the region is a rectangle, integration limits are simple to interchange.)

$$\iiint_R f \, dv = \int_0^1 \int_0^2 xy \Big(\frac{z^3}{3} \Big|_0^3 \Big) \, dy \, dx = \frac{27}{3} \int_0^1 \int_0^2 xy \, dy \, dx.$$

$$\iiint_R f \, dv = 9 \int_0^1 x \Big(\frac{y^2}{2} \Big|_0^2 \Big) \, dx = 18 \int_0^1 x \, dx = 9.$$

We conclude: $\displaystyle \iiint_R f \, dv = 9.$ ◁

Triple integrals in Cartesian coordinates (Sect. 15.5)

- Triple integrals in rectangular boxes.
- **Triple integrals in arbitrary domains.**
- Volume on a region in space.

Triple integrals in arbitrary domains

Theorem

If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the domain

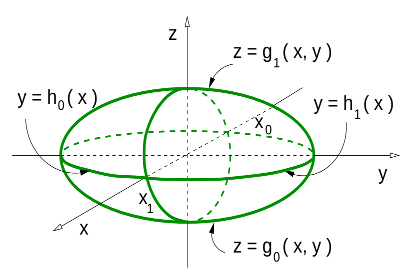
$$D = \big\{x \in [x_0, x_1],\ y \in [h_0(x), h_1(x)],\ z \in [g_0(x,y), g_1(x,y)]\big\},$$

where $g_0, g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ are continuous, then the triple integral of the function f in the region D is given by

$$\iiint_D f\,dv = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Example

In the case that D is an ellipsoid, the figure represents the graph of functions g_1, g_0 and h_1, h_0.



Triple integrals in Cartesian coordinates (Sect. 15.5)

- Triple integrals in rectangular boxes.
- Triple integrals in arbitrary domains.
- **Volume on a region in space.**

Volume on a region in space

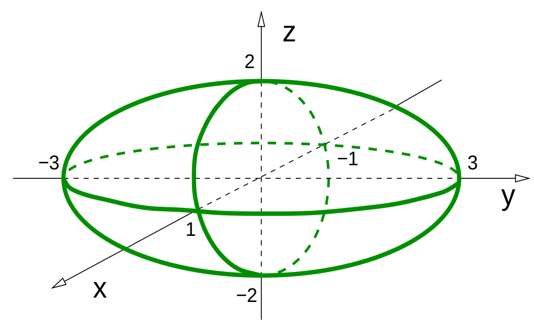
Remark: The volume of a bounded, closed region $D \in \mathbb{R}^3$ is

$$V = \iiint_D dv.$$

Example

Find the integration limits needed to compute the volume of the ellipsoid $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$.

Solution: We first sketch the integration domain.



Volume on a region in space

Example

Find the integration limits needed to compute the volume of the ellipsoid $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$.

Solution: The functions $z = g_1$ and $z = g_0$ are, respectively,

$$z = 2\sqrt{1 - x^2 - \frac{y^2}{3^2}}, \quad z = -2\sqrt{1 - x^2 - \frac{y^2}{3^2}}.$$

The functions $y = h_1$ and $y = h_0$ are defined on $z = 0$, and are given by, $y = 3\sqrt{1 - x^2}$ and $y = -3\sqrt{1 - x^2}$, respectively.

The limits on x are defined at $z = 0$, $y = 0$: $x = \pm 1$. Hence,

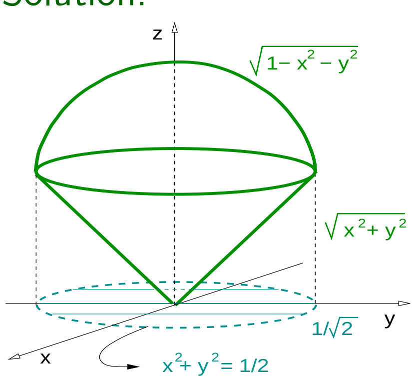
$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx. \qquad \triangleleft$$

Volume on a region in space

Example

Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution:



The top surface is the sphere,

$$z = \sqrt{1 - x^2 - y^2}.$$

The bottom surface is the cone,

$$z = \sqrt{x^2 + y^2}.$$

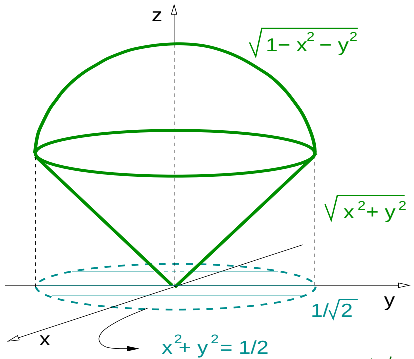
The limits on y are obtained projecting the 3-dimensional figure onto the plane $z = 0$. We obtain the disk $x^2 + y^2 = 1/2$.
(The polar radius at the intersection cone-sphere was $r_0 = 1/\sqrt{2}$.)

Volume on a region in space

Example

Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: Recall: $z = \sqrt{1 - x^2 - y^2}$, $z = \sqrt{x^2 + y^2}$.



The y-top of the disk is,

$$y = \sqrt{1/2 - x^2}.$$

The y-bottom of the disk is,

$$y = -\sqrt{1/2 - x^2}.$$

We conclude: $V = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$ $\quad \triangleleft$

Volume on a region in space

Example

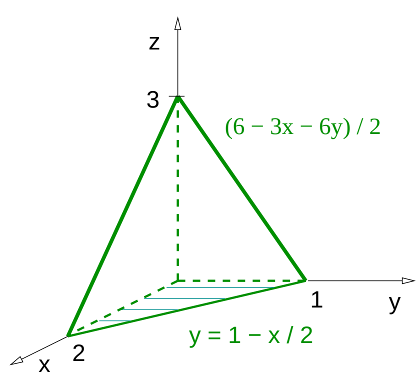
Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution:

The region is given by the first octant and below the plane

$$3x + 6y + 2z = 6.$$

This plane contains the points $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$.



In z the limits are $z = (6 - 3x - 6y)/2$ and $z = 0$.

Volume on a region in space

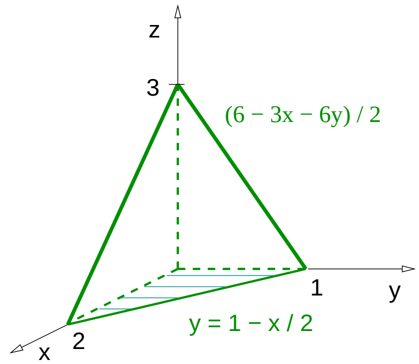
Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution: In z the limits are $z = (6 - 3x - 6y)/2$ and $z = 0$.

At $z = 0$ the projection of the region is the triangle $x \geqslant 0$, $y \geqslant 0$, and $x + 2y \leqslant 2$.

In y the limits are $y = 1 - x/2$ and $y = 0$.



We conclude: $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

Volume on a region in space

Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution: Recall: $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

$$V = 3 \int_0^2 \int_0^{1-x/2} \Big(1 - \frac{x}{2} - y\Big)\, dy\, dx,$$

$$V = 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(y\Big|_0^{(1-x/2)}\Big) - \Big(\frac{y^2}{2}\Big|_0^{(1-x/2)}\Big)\Big]\, dx,$$

$$V = 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(1 - \frac{x}{2}\Big) - \frac{1}{2}\Big(1 - \frac{x}{2}\Big)^2\Big]\, dx.$$

We only need to compute: $V = \dfrac{3}{2} \int_0^2 \Big(1 - \dfrac{x}{2}\Big)^2 dx.$

Volume on a region in space

Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution: Recall: $V = \frac{3}{2}\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx.$

Substitute $u = 1 - x/2$, then $du = -dx/2$, so

$$V = -3\int_1^0 u^2 du = 3\int_0^1 u^2\, du = 3\left(\frac{u^3}{3}\Big|_0^1\right)$$

We conclude: $V = 1$. $\triangleleft$

Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x, y, z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

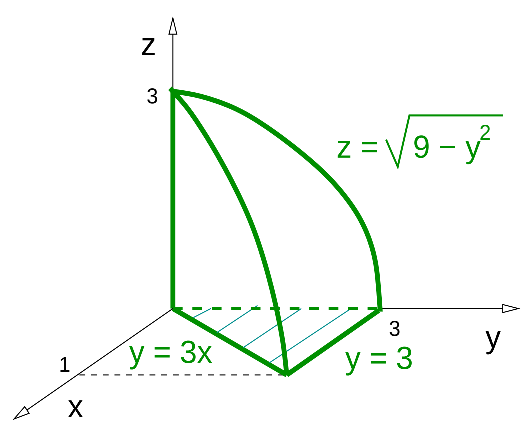
Solution:

The upper surface is

$$z = \sqrt{9 - y^2},$$

the bottom surface is

$$z = 0.$$



The y coordinate is bounded below by the line $y = 3x$ and above by $y = 3$. (Because of the cylinder equation at $z = 0$.)

Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x,y,z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

Solution: Recall: $0 \leqslant z \leqslant \sqrt{9-y^2}$ $3x \leqslant y \leqslant 3$, and $f = z$.

$$I = \iiint_D f\,dv = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx,$$

$$I = \int_0^1 \int_{3x}^3 \Big(\frac{z^2}{2}\Big|_0^{\sqrt{9-y^2}} \Big)\,dy\,dx,$$

$$I = \frac{1}{2}\int_0^1 \int_{3x}^3 (9-y^2)dy\,dx,$$

$$I = \frac{1}{2}\int_0^1 \Big[27(1-x) - \Big(\frac{y^3}{3}\Big|_{3x}^3 \Big) \Big]\,dx.$$

Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x,y,z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

Solution: Recall: $I = \frac{1}{2}\int_0^1 \Big[27(1-x) - \Big(\frac{y^3}{3}\Big|_{3x}^3 \Big) \Big]\,dx$. So,

$$I = \frac{1}{2}\int_0^1 \Big[27(1-x) - 9(1-x)^3 \Big]\,dx,$$

$$I = \frac{9}{2}\int_0^1 \Big[3(1-x) - (1-x)^3 \Big]\,dx.$$

Substitute $u = 1 - x$, then $du = -dx$, so,

$$I = \frac{9}{2}\int_0^1 (3u - u^3)du.$$

Triple integrals in arbitrary domains

Example

Compute the triple integral of $f(x, y, z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

Solution: Recall: $I = \frac{9}{2}\int_0^1 (3u - u^3)\,du.$

$$I = \frac{9}{2}\Big[\frac{3}{2}\Big(u^2\Big|_0^1\Big) - \frac{1}{4}\Big(u^4\Big|_0^1\Big)\Big],$$

$$I = \frac{9}{2}\Big(\frac{3}{2} - \frac{1}{4}\Big).$$

We conclude: $\iiint_D f\,dv = \frac{45}{8}.$ $\triangleleft$