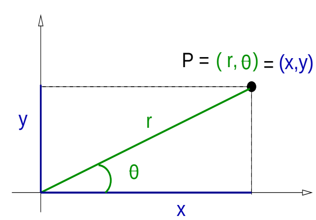
Double integrals in polar coordinates (Sect. 15.4)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.

Review: Polar coordinates

Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ) defined by the picture.



Theorem (Cartesian-polar transformations)

The Cartesian coordinates of a point $P = (r, \theta)$ are given by

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

The polar coordinates of a point $P = (x, y)$ in the first or fourth quadrants are given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

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Double integrals on disk sections

Theorem

If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in the region

$$R = \{(r,\theta) \in \mathbb{R}^2 : r \in [r_0, r_1],\ \theta \in [\theta_0, \theta_1]\}$$

where $0 \leqslant \theta_0 \leqslant \theta_1 \leqslant 2\pi$, then the double integral of function f in that region can be expressed in polar coordinates as follows,

$$\iint_R f\,dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r,\theta)\, r\, dr\, d\theta.$$

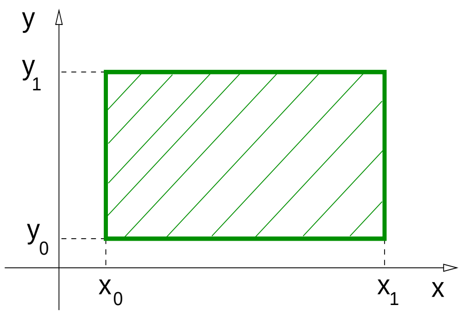
Remark:

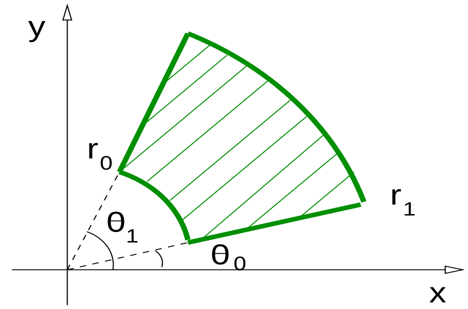
- Disk sections in polar coordinates are the analogous to rectangular sections in Cartesian coordinates.
- The boundaries of each domain, a rectangle in Cartesian and a disk section in polar coordinates, are defined by a constant value of a coordinate.
- Notice the extra factor r on the right-hand side above.

Double integrals on disk sections

Remark:

Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.





$$x_0 \leqslant x \leqslant x_1,$$
$$y_0 \leqslant y \leqslant y_1.$$

$$r_0 \leqslant r \leqslant r_1,$$
$$\theta_0 \leqslant \theta \leqslant \theta_1.$$

Double integrals on disk sections

Example

Find the area of an arbitrary circular section
$R = \{(r, \theta) \in \mathbb{R}^2 \ : \ r \in [r_0, r_1], \ \theta \in [\theta_0, \theta_1]\}$.
Evaluate that area in the particular case of a disk with radius R.

Solution:

$$A = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} (r\, dr)\, d\theta = \int_{\theta_0}^{\theta_1} \Big(\frac{r^2}{2}\Big|_{r_0}^{r_1}\Big)\, d\theta,$$

$$A = \int_{\theta_0}^{\theta_1} \frac{1}{2}\big[(r_1)^2 - (r_0)^2\big]\, d\theta \quad \Rightarrow \quad A = \frac{1}{2}\big[(r_1)^2 - (r_0)^2\big](\theta_1 - \theta_0).$$

The case of a disk is: $\theta_0 = 0$, $\theta_1 = 2\pi$, $r_0 = 0$ and $r_1 = R$.

In that case we re-obtain the usual formula $A = \pi R^2$. $\triangleleft$

Double integrals on disk sections

Example

Find the integral of $f(r,\theta) = r^2\cos(\theta)$ in the disk
$R = \{(r,\theta) \in \mathbb{R}^2 \;:\; r \in [0,1],\; \theta \in [0,\pi/4]\}$.

Solution:

$$\iint_R f\,dA = \int_0^{\pi/4}\int_0^1 r^2\cos(\theta)(r\,dr)\,d\theta,$$

$$\iint_R f\,dA = \int_0^{\pi/4}\Big(\frac{r^4}{4}\Big|_0^1\Big)\cos(\theta)\,d\theta = \frac{1}{4}\sin(\theta)\Big|_0^{\pi/4}.$$

We conclude that $\iint_R f\,dA = \sqrt{2}/8$. $\triangleleft$

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Double integrals in arbitrary regions

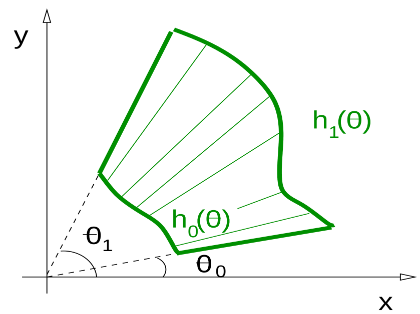
Theorem

If the function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in the region

$$R = \{(r,\theta) \in \mathbb{R}^2 \ : \ r \in [h_0(\theta), h_1(\theta)], \ \theta \in [\theta_0, \theta_1]\}.$$

where $0 \leqslant h_0(\theta) \leqslant h_1(\theta)$ are continuous functions defined on an interval $[\theta_0, \theta_1]$, then the integral of function f in R is given by

$$\iint_R f(r,\theta)\, dA = \int_{\theta_0}^{\theta_1} \int_{h_0(\theta)}^{h_1(\theta)} f(r,\theta) r\, dr\, d\theta.$$



Double integrals in arbitrary regions

Example

Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: We first show that these curves are actually circles.

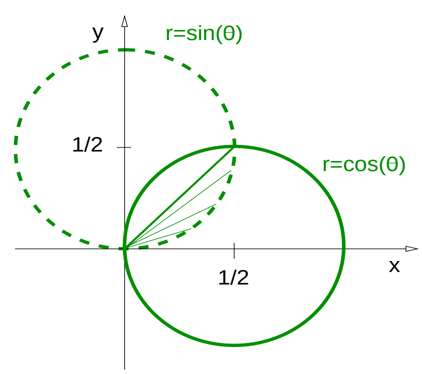
$$r = \cos(\theta) \quad \Leftrightarrow \quad r^2 = r\cos(\theta) \quad \Leftrightarrow \quad x^2 + y^2 = x.$$

Completing the square in x we obtain

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$$

Analogously, $r = \sin(\theta)$ is the circle

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$



Double integrals in arbitrary regions.

Example

Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: $A = 2 \int_0^{\pi/4} \int_0^{\sin(\theta)} r\, dr\, d\theta = 2 \int_0^{\pi/4} \frac{1}{2} \sin^2(\theta)\, d\theta;$

$$A = \int_0^{\pi/4} \frac{1}{2} \big[1 - \cos(2\theta)\big]\, d\theta = \frac{1}{2} \Big[\Big(\frac{\pi}{4} - 0\Big) - \frac{1}{2} \sin(2\theta)\Big|_0^{\pi/4} \Big];$$

$$A = \frac{1}{2} \Big[\frac{\pi}{4} - \Big(\frac{1}{2} - 0\Big) \Big] = \frac{\pi}{8} - \frac{1}{4} \quad \Rightarrow \quad A = \frac{1}{8}(\pi - 2).$$

$\triangleleft$

Also works: $A = \int_0^{\pi/4} \int_0^{\sin(\theta)} r\, dr\, d\theta + \int_{\pi/4}^{\pi/2} \int_0^{\cos(\theta)} r\, dr\, d\theta.$

Double integrals in polar coordinates (Sect. 15.4)

- Review: Polar coordinates.
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- **Changing Cartesian integrals into polar integrals.**
- Computing volumes using double integrals.

Changing Cartesian integrals into polar integrals

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and $f(x, y)$ represents the function values in Cartesian coordinates, then holds

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(r\cos(\theta), r\sin(\theta))r\,dr\,d\theta.$$

Example

Compute the integral of $f(x,y) = x^2 + 2y^2$ on
$D = \{(x,y) \in \mathbb{R}^2 \;:\; 0 \leqslant y, \quad 0 \leqslant x, \quad 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Solution: First, transform Cartesian into polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$. Since $f(x,y) = (x^2 + y^2) + y^2$,

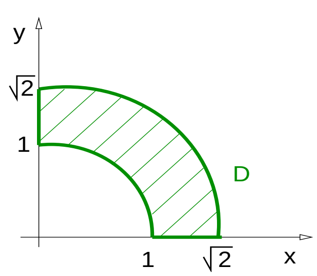
$$f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\sin^2(\theta).$$

Changing Cartesian integrals into polar integrals

Example

Compute the integral of $f(x,y) = x^2 + 2y^2$ on
$D = \{(x,y) \in \mathbb{R}^2 \;:\; 0 \leqslant y,\; 0 \leqslant x, \quad 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Solution: We computed: $f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\sin^2(\theta)$.



The region is

$$D = \left\{(r,\theta) \in \mathbb{R}^2 : 0 \leqslant \theta \leqslant \frac{\pi}{2},\; 1 \leqslant r \leqslant \sqrt{2}\right\}$$

$$\iint_D f(r,\theta)dA = \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2\big(1 + \sin^2(\theta)\big)r\,drd\theta,$$

$$\iint_D f(r,\theta)dA = \left[\int_0^{\pi/2} \big(1 + \sin^2(\theta)\big)\,d\theta\right]\left[\int_1^{\sqrt{2}} r^3 dr\right].$$

Changing Cartesian integrals into polar integrals

Example

Compute the integral of $f(x, y) = x^2 + 2y^2$ on
$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant y, \ 0 \leqslant x, \quad 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Solution: $\displaystyle \iint_D f(r, \theta) dA = \left[\int_0^{\pi/2} \left(1 + \sin^2(\theta)\right) d\theta\right] \left[\int_1^{\sqrt{2}} r^3 dr\right]$.

$$\iint_D f(r, \theta) dA = \left[\left(\theta\Big|_0^{\pi/2}\right) + \int_0^{\pi/2} \frac{1}{2}\left(1 - \cos(2\theta)\right) d\theta\right] \frac{1}{4}\left(r^4\Big|_1^{\sqrt{2}}\right)$$

$$\iint_D f(r, \theta) dA = \left[\frac{\pi}{2} + \frac{1}{2}\left(\theta\Big|_0^{\pi/2}\right) - \frac{1}{4}\left(\sin(2\theta)\Big|_0^{\pi/2}\right)\right] \frac{3}{4} = \left[\frac{\pi}{2} + \frac{\pi}{4}\right] \frac{3}{4}.$$

We conclude: $\displaystyle \iint_D f(r, \theta) dA = \frac{9}{16}\pi.$ ◁

Changing Cartesian integrals into polar integrals

Example

Integrate $f(x, y) = e^{-(x^2+y^2)}$ on the domain
$D = \{(r, \theta) \in R^2 \ : \ 0 \leqslant \theta \leqslant \pi, \ 0 \leqslant r \leqslant 2\}$.

Solution: Since $f(r\cos(\theta), r\sin(\theta)) = e^{-r^2}$, the double integral is

$$\iint_D f(x, y)\, dx\, dy = \int_0^{\pi} \int_0^2 e^{-r^2} r\, dr\, d\theta.$$

Substitute $u = r^2$, hence $du = 2r\, dr$, we obtain

$$\iint_D f(x, y)\, dx\, dy = \frac{1}{2} \int_0^{\pi} \int_0^4 e^{-u} du\, d\theta = \frac{1}{2} \int_0^{\pi} \left(-e^{-u}\Big|_0^4\right) d\theta.$$

We conclude: $\displaystyle \iint_D f(x, y)\, dx\, dy = \frac{\pi}{2}\left(1 - \frac{1}{e^4}\right).$ ◁

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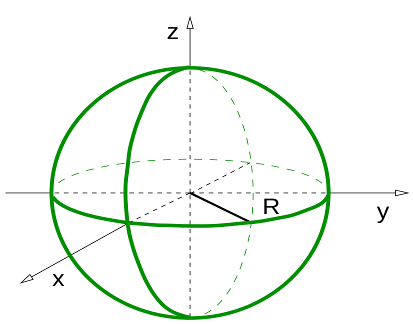
Computing volumes using double integrals

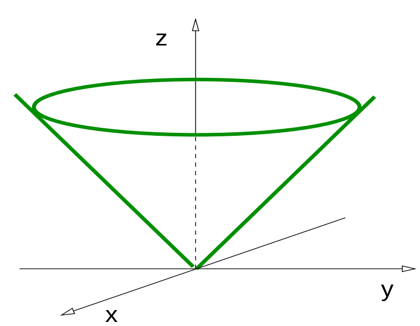
Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: Let us first draw the sets that form the volume we are interested to compute.



$$z = \pm\sqrt{1 - r^2},$$



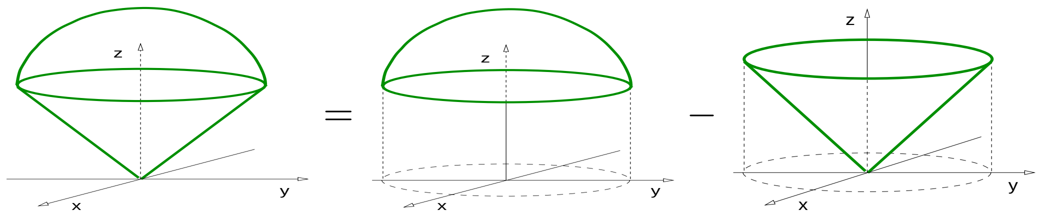
$$z = r.$$

Computing volumes using double integrals

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: The integration region can be decomposed as follows:



The volume we are interested to compute is:

$$V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2}\,(rdr)d\theta - \int_0^{2\pi} \int_0^{r_0} r\,(rdr)d\theta.$$

We need to find r_0, the intersection of the cone and the sphere.

Computing volumes using double integrals

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: We find r_0, the intersection of the cone and the sphere.

$$\sqrt{1 - r_0^2} = r_0 \quad \Leftrightarrow \quad 1 - r_0^2 = r_0^2 \quad \Leftrightarrow \quad 2r_0^2 = 1;$$

that is, $r_0 = 1/\sqrt{2}$. Therefore

$$V = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\,(r\,dr)d\theta - \int_0^{2\pi} \int_0^{1/\sqrt{2}} r\,(r\,dr)d\theta.$$

$$V = 2\pi \Big[\int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\,(r\,dr) - \int_0^{1/\sqrt{2}} r\,(r\,dr) \Big].$$

Computing volumes using double integrals

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: $V = 2\pi \Big[\int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\,(r\,dr) - \int_0^{1/\sqrt{2}} r\,(r\,dr) \Big]$.

Use the substitution $u = 1 - r^2$, so $du = -2r\,dr$. We obtain,

$$V = 2\pi \Big[\frac{1}{2} \int_{1/2}^{1} u^{1/2}\,du - \frac{1}{3} r^3 \Big|_0^{1/\sqrt{2}} \Big],$$

$$V = 2\pi \Big[\frac{1}{2}\,\frac{2}{3} u^{3/2} \Big|_{1/2}^{1} - \frac{1}{3}\,\frac{1}{2^{3/2}} \Big] = \frac{2\pi}{3} \Big[1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}} \Big],$$

We conclude: $V = \frac{\pi}{3}\big(2 - \sqrt{2}\big)$. $\triangleleft$