Double integrals in polar coordinates (Sect. 15.4)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.

Review: Polar coordinates

Definition
The polar coordinates of a point \( P \in \mathbb{R}^2 \) is the ordered pair \((r, \theta)\) defined by the picture.

Theorem (Cartesian-polar transformations)
The Cartesian coordinates of a point \( P = (r, \theta) \) are given by
\[
x = r \cos(\theta), \quad y = r \sin(\theta).
\]

The polar coordinates of a point \( P = (x, y) \) in the first or fourth quadrants are given by
\[
r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).
\]
Double integrals on disk sections

Theorem

If \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in the region
\[
R = \{(r, \theta) \in \mathbb{R}^2 : r \in [r_0, r_1], \ \theta \in [\theta_0, \theta_1]\}
\]

where \( 0 \leq \theta_0 \leq \theta_1 \leq 2\pi \), then the double integral of function \( f \) in that region can be expressed in polar coordinates as follows,
\[
\int \int_R f \, dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r, \theta) \, r \, dr \, d\theta.
\]

Remark:

- Disk sections in polar coordinates are the analogous to rectangular sections in Cartesian coordinates.
- The boundaries of each domain, a rectangle in Cartesian and a disk section in polar coordinates, are defined by a constant value of a coordinate.
- Notice the extra factor \( r \) on the right-hand side above.
Double integrals on disk sections

Remark:
Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.

Example
Find the area of an arbitrary circular section
\( R = \{(r, \theta) \in \mathbb{R}^2 : r \in [r_0, r_1], \ \theta \in [\theta_0, \theta_1]\} \).
Evaluate that area in the particular case of a disk with radius \( R \).

Solution:
\[
A = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} (r \, dr) \, d\theta = \int_{\theta_0}^{\theta_1} \left( \frac{r^2}{2} \bigg|_{r_0}^{r_1} \right) \, d\theta,
\]
\[
A = \int_{\theta_0}^{\theta_1} \frac{1}{2} [(r_1)^2 - (r_0)^2] \, d\theta \quad \Rightarrow \quad A = \frac{1}{2} [(r_1)^2 - (r_0)^2] (\theta_1 - \theta_0).
\]
The case of a disk is: \( \theta_0 = 0, \ \theta_1 = 2\pi, \ r_0 = 0 \) and \( r_1 = R \).
In that case we re-obtain the usual formula \( A = \pi R^2 \).
Example
Find the integral of \( f(r, \theta) = r^2 \cos(\theta) \) in the disk \( R = \{(r, \theta) \in \mathbb{R}^2 : r \in [0, 1], \ \theta \in [0, \pi/4]\} \).

Solution:
\[
\iint_R f \, dA = \int_0^{\pi/4} \int_0^1 r^2 \cos(\theta) (r \, dr) \, d\theta,
\]
\[
\iint_R f \, dA = \int_0^{\pi/4} \left( \frac{r^4}{4} \right)_0^1 \cos(\theta) \, d\theta = \frac{1}{4} \sin(\theta) \bigg|_0^{\pi/4}.
\]

We conclude that \( \iint_R f \, dA = \sqrt{2}/8 \). \(\triangledown\)

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Double integrals in arbitrary regions

Theorem

If the function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in the region

$$R = \{(r, \theta) \in \mathbb{R}^2 : r \in [h_0(\theta), h_1(\theta)], \ \theta \in [\theta_0, \theta_1]\}.$$ 

where $0 \leq h_0(\theta) \leq h_1(\theta)$ are continuous functions defined on an interval $[\theta_0, \theta_1]$, then the integral of function $f$ in $R$ is given by

$$\int \int_R f(r, \theta) \, dA = \int_{\theta_0}^{\theta_1} \int_{h_0(\theta)}^{h_1(\theta)} f(r, \theta) r \, dr \, d\theta.$$ 

Double integrals in arbitrary regions

Example

Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: We first show that these curves are actually circles.

$$r = \cos(\theta) \iff r^2 = r \cos(\theta) \iff x^2 + y^2 = x.$$ 

Completing the square in $x$ we obtain

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$$ 

Analogously, $r = \sin(\theta)$ is the circle

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$
Double integrals in arbitrary regions.

Example
Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: $A = 2 \int_{0}^{\pi/4} \int_{0}^{\sin(\theta)} r \, dr \, d\theta = 2 \int_{0}^{\pi/4} \frac{1}{2} \sin^2(\theta) \, d\theta$;

$A = \int_{0}^{\pi/4} \frac{1}{2} [1 - \cos(2\theta)] \, d\theta = \frac{1}{2} \left[ \left( \frac{\pi}{4} - 0 \right) - \frac{1}{2} \sin(2\theta) \right]_{0}^{\pi/4}$;

$A = \frac{1}{2} \left[ \frac{\pi}{4} - \left( \frac{1}{2} - 0 \right) \right] = \frac{\pi}{8} - \frac{1}{4} \Rightarrow A = \frac{1}{8} (\pi - 2)$.

Also works: $A = \int_{0}^{\pi/4} \int_{0}^{\sin(\theta)} r \, dr \, d\theta + \int_{\pi/4}^{\pi/2} \int_{0}^{\cos(\theta)} r \, dr \, d\theta$.

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Changing Cartesian integrals into polar integrals

**Theorem**

If \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) is a continuous function, and \( f(x, y) \) represents the function values in Cartesian coordinates, then holds

\[
\int\int_D f(x, y) \, dx \, dy = \int\int_D f(r \cos(\theta), r \sin(\theta)) \, r \, dr \, d\theta.
\]

**Example**

Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on \( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2\} \).

**Solution:** First, transform Cartesian into polar coordinates: \( x = r \cos(\theta), \ y = r \sin(\theta) \). Since \( f(x, y) = (x^2 + y^2) + y^2 \),

\[
f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \sin^2(\theta).
\]

The region is

\[
D = \left\{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \frac{\pi}{2}, \ 1 \leq r \leq \sqrt{2}\right\}
\]

\[
\int\int_D f(r, \theta) \, dA = \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2(1 + \sin^2(\theta)) \, r \, dr \, d\theta,
\]

\[
\int\int_D f(r, \theta) \, dA = \left[ \int_0^{\pi/2} (1 + \sin^2(\theta)) \, d\theta \right] \left[ \int_1^{\sqrt{2}} r^3 \, dr \right].
\]
Changing Cartesian integrals into polar integrals

Example

Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on 
\( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ \ 1 \leq x^2 + y^2 \leq 2\} \).

Solution: \[
\int_0^\pi \int_1^{\sqrt{2}} \left( \frac{\pi}{2} + \frac{1}{2} \left( \int_0^{\pi/2} \frac{1}{4} \sin(2\theta) d\theta \right) \right) \frac{3}{4} = \int_0^\pi \int_1^{\sqrt{2}} \left( \frac{\pi}{2} + \frac{\pi}{4} \right) \frac{3}{4}.
\]

We conclude: \[
\int_0^\pi \int_1^{\sqrt{2}} f(r, \theta) dA = \frac{9}{16} \pi.
\]

Changing Cartesian integrals into polar integrals

Example

Integrate \( f(x, y) = e^{-x^2-y^2} \) on the domain 
\( D = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2\} \).

Solution: Since \( f(r \cos(\theta), r \sin(\theta)) = e^{-r^2} \), the double integral is
\[
\int_0^\pi \int_0^2 e^{-r^2} r \, dr \, d\theta.
\]
Substitute \( u = r^2 \), hence \( du = 2r \, dr \), we obtain
\[
\int_0^\pi \int_0^4 e^{-u} du \, d\theta = \frac{1}{2} \int_0^\pi \left( -e^{-u} \right)_0^4 \, d\theta.
\]
We conclude: \[
\int_0^\pi \int_0^2 f(x, y) \, dx \, dy = \frac{\pi}{2} \left( 1 - \frac{1}{e^4} \right).
\]

◁
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- **Computing volumes using double integrals.**

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**Example**

Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

**Solution:** Let us first draw the sets that form the volume we are interested to compute.

\[
z = \pm \sqrt{1 - r^2},
\]

\[
z = r.
\]
Computing volumes using double integrals

Example
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: The integration region can be decomposed as follows:

The volume we are interested to compute is:

\[
V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2} (r dr) d\theta - \int_0^{2\pi} \int_0^{r_0} r (r dr) d\theta.
\]

We need to find \( r_0 \), the intersection of the cone and the sphere.

\[
\sqrt{1 - r_0^2} = r_0 \quad \Rightarrow \quad 1 - r_0^2 = r_0^2 \quad \Rightarrow \quad 2r_0^2 = 1;
\]

that is, \( r_0 = 1/\sqrt{2} \). Therefore

\[
V = 2\pi \left[ \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r dr) - \int_0^{1/\sqrt{2}} r (r dr) \right].
\]
Computing volumes using double integrals

Example
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: \( V = 2\pi \left[ \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r \, dr) - \int_0^{1/\sqrt{2}} r (r \, dr) \right] \).

Use the substitution \( u = 1 - r^2 \), so \( du = -2r \, dr \). We obtain,

\[
V = 2\pi \left[ \frac{1}{2} \int_{1/2}^{1} u^{1/2} \, du - \frac{1}{3} r^3 \bigg|_0^{1/\sqrt{2}} \right],
\]

\[
V = 2\pi \left[ \frac{1}{2} \frac{2}{3} u^{3/2} \bigg|_{1/2}^{1} - \frac{1}{3} \left( \frac{1}{23/2} - \frac{1}{23/2} \right) \right] = \frac{2\pi}{3} \left[ 1 - \frac{1}{23/2} - \frac{1}{23/2} \right],
\]

We conclude: \( V = \frac{\pi}{3} \left( 2 - \sqrt{2} \right) \). ◀