Areas and double integrals. (Sect. 15.3)

- Areas of a region on a plane.
- Average value of a function.
- More examples of double integrals.

Areas of a region on a plane

Definition

The *area* of a closed, bounded region R on a plane is given by

$$A = \iint_R dx\, dy.$$

Remark:

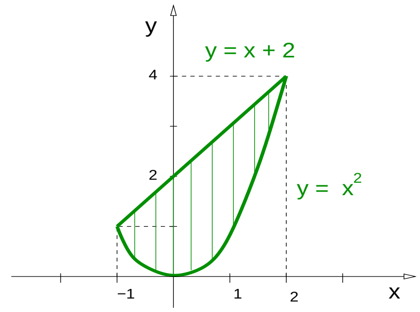
- To compute the area of a region R we integrate the function $f(x, y) = 1$ on that region R.
- The area of a region R is computed as the volume of a 3-dimensional region with base R and height equal to 1.

Areas of a region on a plane

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [-1, 2], \ y \in [x^2, x + 2]\}$.

Solution: We express the region R as an integral Type I, integrating first on vertical directions:



$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$

$$A = \int_{-1}^{2} \Big(y \Big|_{x^2}^{x+2} \Big)\, dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \Big(\frac{x^2}{2} + 2x - \frac{x^3}{3} \Big) \Big|_{-1}^{2}.$$

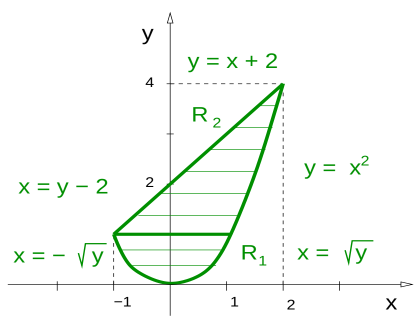
$$A = 2 - \frac{1}{2} + 4 + 2 - \frac{8}{3} - \frac{1}{3} = 8 - \frac{1}{2} - 3 \quad \Rightarrow \quad A = \frac{9}{2}. \quad \triangleleft$$

Areas of a region on a plane

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: We express the region R as an integral Type II, integrating first on horizontal directions:



$$A = \iint_{R_1} dx\, dy + \iint_{R_2} dx\, dy.$$

$$A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} dx\, dy.$$

We must get the same result: $A = 9/2$.

Areas of a region on a plane

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: Recall: $A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy.$

$$A = \int_0^1 2\sqrt{y}\,dy + \int_1^4 \big(\sqrt{y} - y + 2\big)\,dy$$

$$A = 2\Big(\frac{2}{3}\,y^{3/2}\Big)\Big|_0^1 + \Big(\frac{2}{3}\,y^{3/2} - \frac{y^2}{2} + 2y\Big)\Big|_1^4$$

$$A = \frac{4}{3} + \frac{16}{3} - \frac{2}{3} - 8 + \frac{1}{2} + 8 - 2 = 6 - \frac{3}{2}.$$

We conclude that $A = \dfrac{9}{2}$. ◁

Areas and double integrals. (Sect. 15.3)

- Areas of a region on a plane.
- **Average value of a function.**
- More examples of double integrals.

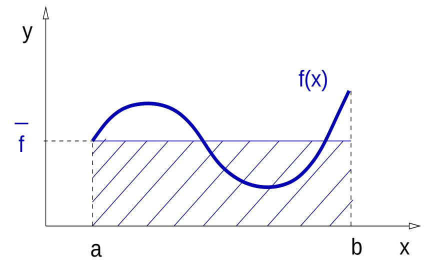
Average value of a function

Review: The average of a single variable function.

Definition

The *average* of a function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{(b-a)} \int_a^b f(x)\, dx.$$



Definition

The *average* of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on the region R with area $A(R)$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{A(R)} \iint_R f(x,y)\, dx\, dy.$$

Average value of a function

Example

Find the average of $f(x,y) = xy$ on the region $R = \{(x,y) \in \mathbb{R}^2 \ : \ x \in [0,2],\ y \in [0,3]\}$.

Solution: The area of the rectangle R is $A(R) = 6$.

We only need to compute $I = \iint_R f(x,y)\, dx\, dy$.

$$I = \int_0^2 \int_0^3 xy\, dy\, dx = \int_0^2 x\Big(\frac{y^2}{2}\Big|_0^3\Big)\, dx = \int_0^2 \frac{9}{2}x\, dx.$$

$$I = \frac{9}{2}\Big(\frac{x^2}{2}\Big|_0^2\Big) \quad \Rightarrow \quad I = 9.$$

Since $\overline{f} = I/A(R) = 9/6$, we get $\overline{f} = 3/2$. $\triangleleft$

Areas and double integrals. (Sect. 15.3)

- Areas of a region on a plane.
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- **More examples of double integrals.**

More examples of double integrals

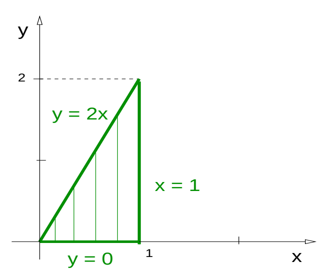
Example

Find the integral of $\rho(x, y) = x + y$ in the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$.

Solution: We need to compute

$$M = \iint_R \rho(x, y)\, dxdy.$$

Remark: If ρ is the mass density, then M is the total mass.



$$M = \int_0^1 \int_0^{2x} (x + y)\, dy\, dx = \int_0^1 \Big[x \Big(y\Big|_0^{2x} \Big) + \Big(\frac{y^2}{2}\Big|_0^{2x} \Big) \Big]\, dx.$$

$$M = \int_0^1 \big[2x^2 + 2x^2 \big]\, dx = 4\, \frac{x^3}{3}\Big|_0^1 \quad \Rightarrow \quad M = \frac{4}{3}. \qquad \triangleleft$$

More examples of double integrals

Example

Given the function $\rho(x, y) = x + y$, the number M computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

$$\bar{r}_x = \frac{1}{M} \int_R x\rho(x, y)\, dy\, dx, \quad \bar{r}_y = \frac{1}{M} \iint_R y\rho(x, y)\, dy\, dx.$$

Remark: $\mathbf{r} = \langle \bar{r}_x, \bar{r}_y \rangle$ is the center of mass of the body.

Solution: Recall: $M = \frac{4}{3}$. We need to compute

$$\bar{r}_x = \frac{1}{M} \int_0^1 \int_0^{2x} (x+y)x\, dydx = \frac{3}{4} \int_0^1 \left[x^2 \left(y \Big|_0^{2x} \right) + x \left(\frac{y^2}{2} \Big|_0^{2x} \right) \right] dx$$

$$\bar{r}_x = \frac{3}{4} \int_0^1 \left[2x^3 + 2x^3 \right] dx = \frac{3}{4}\, x^4 \Big|_0^1 \quad \Rightarrow \quad \bar{r}_x = \frac{3}{4}.$$

More examples of double integrals

Example

Given the function $\rho(x, y) = x + y$, the number M computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

$$\bar{r}_x = \frac{1}{M} \int_R x\rho(x, y)\, dy\, dx, \quad \bar{r}_y = \frac{1}{M} \iint_R y\rho(x, y)\, dy\, dx.$$

Solution: Recall: $M = \frac{4}{3}$ and $\bar{r}_x = \frac{3}{4}$.

$$\bar{r}_y = \frac{1}{M} \int_0^1 \int_0^{2x} (x+y)y\, dydx = \frac{3}{4} \int_0^1 \left[x \left(\frac{y^2}{2} \Big|_0^{2x} \right) + \left(\frac{y^3}{3} \Big|_0^{2x} \right) \right] dx$$

$$\bar{r}_y = \frac{3}{4} \int_0^1 \left[2x^3 + \frac{8}{3} x^3 \right] dx = \frac{3}{4} \left[2 \left(\frac{x^4}{4} \Big|_0^1 \right) + \frac{8}{3} \left(\frac{x^4}{4} \Big|_0^1 \right) \right]$$

$$\bar{r}_y = \frac{3}{4} \left[\frac{1}{2} + \frac{2}{3} \right] = \frac{3}{4}\, \frac{7}{6} \Rightarrow \bar{r}_y = \frac{7}{8}.$$

$\triangleleft$

More examples of double integrals

Definition

The *centroid* of a region R in the plane is the vector $\mathbf{c}$ given by

$$\mathbf{c} = \frac{1}{A(R)} \iint_R \langle x, y\rangle \, dx\, dy, \quad \text{where} \quad A(R) = \iint_R dx\, dy.$$

Remark:

- The centroid of a region can be seen as the center of mass vector of that region in the case that the mass density is constant.
- When the mass density is constant, it cancels out from the numerator and denominator of the center of mass.

More examples of double integrals

Example

Find the centroid of the triangle inside $y = 0$, $x = 1$ and $y = 2x$.

Solution: The area of the triangle is

$$A(R) = \int_0^1 \int_0^{2x} dy\, dx = \int_0^1 2x\, dx = x^2\Big|_0^1 \quad \Rightarrow \quad A(R) = 1.$$

Therefore, the centroid vector components are given by

$$c_x = \int_0^1 \int_0^{2x} x\, dy\, dx = \int_0^1 2x^2\, dx = 2\Big(\frac{x^3}{3}\Big|_0^1\Big) \quad \Rightarrow \quad c_x = \frac{2}{3}.$$

$$c_y = \int_0^1 \int_0^{2x} y\, dy\, dx = \int_0^1 \Big(\frac{y^2}{2}\Big|_0^{2x}\Big)\, dx = \int_0^1 2x^2\, dx = 2\Big(\frac{x^3}{3}\Big|_0^1\Big)$$

so $c_y = \frac{2}{3}$. We conclude, $\mathbf{c} = \frac{2}{3}\langle 1, 1\rangle$. ◁

More examples of double integrals

Remark: The moment of inertia of an object is a measure of the resistance of the object to changes in its rotation along a particular axis of rotation.

Definition

The *moment of inertia* about the x-axis and the y-axis of a region R in the plane having mass density $\rho : R \subset \mathbb{R}^2 \to \mathbb{R}$ are given by, respectively,

$$I_x = \iint_R y^2 \rho(x, y)\, dx\, dy, \qquad I_y = \iint_R x^2 \rho(x, y)\, dx\, dy.$$

If M denotes the total mass of the region, then the *radii of gyration* about the x-axis and the y-axis are given by

$$R_x = \sqrt{I_x/M} \qquad R_y = \sqrt{I_y/M}.$$

The moment of inertia of an object.

Example

Find the moment of inertia and the radius of gyration about the x-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The moment of inertia I_x is given by

$$I_x = \int_0^1 \int_0^{2x} x^2(x + y)\, dy\, dx = \int_0^1 \left[x^3 \left(y\Big|_0^{2x} \right) + x^2 \left(\frac{y^2}{2}\Big|_0^{2x} \right) \right] dx$$

$$I_x = \int_0^1 4x^4\, dx = 4 \left(\frac{x^5}{5}\Big|_0^1 \right) \quad \Rightarrow \quad I_x = \frac{4}{5}.$$

Since the mass of the region is $M = 4/3$, the radius of gyration along the x-axis is $R_x = \sqrt{I_x/M} = \sqrt{\frac{4}{5}\,\frac{3}{4}}$, that is, $R_x = \sqrt{\frac{3}{5}}$. $\lhd$