Double integrals on regions (Sect. 15.2)

- Review: Fubini's Theorem on rectangular domains.
- Fubini's Theorem on non-rectangular domains.
  - Type I: Domain functions $y(x)$.
  - Type II: Domain functions $x(y)$.
- Finding the limits of integration.

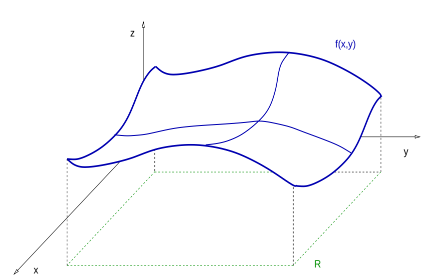
Review: Fubini's Theorem on rectangular domains

Theorem
If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ *is continuous in* $R = [a, b] \times [c, d]$*, then*

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx,$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Remark: Fubini result says that double integrals can be computed doing two one-variable integrals.

Remark: On a rectangle is simple to switch the order of integration in double integrals of continuous functions.



Double integrals on regions (Sect. 15.2)

- Review: Fubini's Theorem on rectangular domains.
- **Fubini's Theorem on non-rectangular domains.**
  - **Type I: Domain functions $y(x)$.**
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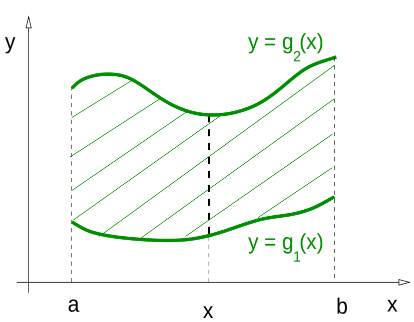
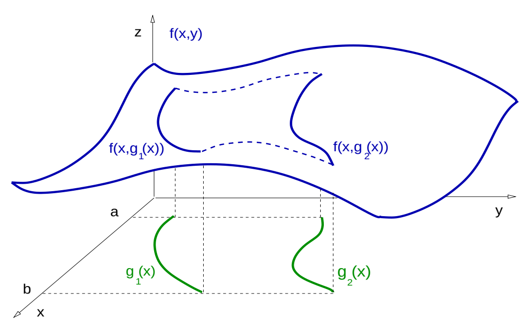
Fubini's Theorem on Type I domains, $y(x)$

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold (Type I):
If $D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ y \in [g_1(x), g_2(x)]\}$, with g_1, g_2 continuous functions on $[a, b]$, then

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$





Double integrals on regions (Sect. 15.2)

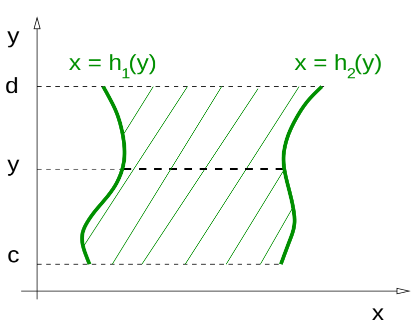
- Review: Fubini's Theorem on rectangular domains.
- **Fubini's Theorem on non-rectangular domains.**
  - Type I: Domain functions $y(x)$.
  - **Type II: Domain functions $x(y)$.**
- Finding the limits of integration.

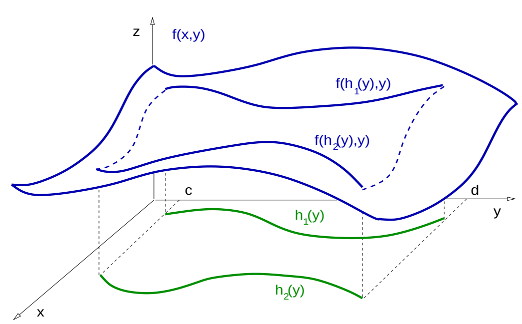
Fubini's Theorem on Type II domains, $x(y)$

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold (Type II):
If $D = \{(x,y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c,d]\}$, with h_1, h_2 continuous functions on $[c,d]$, then

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$





Summary: Fubini's Theorem on non-rectangular domains

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold:

(a) *(Type I) If $D = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ y \in [g_1(x), g_2(x)]\}$, with g_1, g_2 continuous functions on $[a,b]$, then*

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(b) *(Type II) If $D = \{(x,y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c,d]\}$, with h_1, h_2 continuous functions on $[c,d]$, then*

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

A double integral on a Type I domain

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \ :\ 0 \leqslant x \leqslant 1, \quad x^2 \leqslant y \leqslant x\}$.

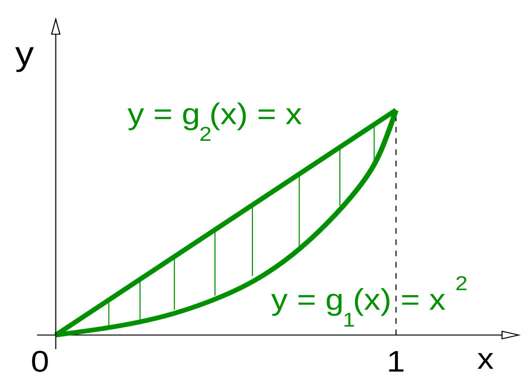
Solution:

This is a Type I domain, with lower boundary

$$y = g_1(x) = x^2,$$

and upper boundary

$$y = g_2(x) = x.$$



A double integral on a Type I domain.

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \; : \; 0 \leqslant x \leqslant 1, \quad x^2 \leqslant y \leqslant x\}$.

Solution: $I = \iint_D f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ with $g_1(x) = x^2$ and $g_2(x) = x$, we obtain

$$I = \iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (x^2 + y^2) dy\,dx,$$

$$I = \int_0^1 \left[x^2 \left(y \Big|_{x^2}^{x} \right) + \left(\frac{y^3}{3} \Big|_{x^2}^{x} \right) \right] dx.$$

$$I = \int_0^1 \left[x^2 \left(x - x^2 \right) + \frac{1}{3} \left(x^3 - x^6 \right) \right] dx.$$

A double integral on a Type I domain

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \; : \; 0 \leqslant x \leqslant 1, \quad x^2 \leqslant y \leqslant x\}$.

Solution: Recall: $I = \int_0^1 \left[x^2 \left(x - x^2 \right) + \frac{1}{3} \left(x^3 - x^6 \right) \right] dx.$

$$I = \int_0^1 \left[x^3 - x^4 + \frac{1}{3} x^3 - \frac{1}{3} x^6 \right] dx = \left[\frac{x^4}{4} - \frac{x^5}{5} + \frac{x^4}{12} - \frac{x^7}{21} \right] \Big|_0^1$$

$$I = \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{(3)(5)(7)}.$$

We conclude: $\iint_D f(x,y)\,dx\,dy = \frac{3}{35}.$ ◁

Summary: Fubini's Theorem on non-rectangular domains

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold:

(a) *(Type I) If $D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ y \in [g_1(x), g_2(x)]\}$, with g_1, g_2 continuous functions on $[a, b]$, then*

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

(b) *(Type II) If $D = \{(x, y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c, d]\}$, with h_1, h_2 continuous functions on $[c, d]$, then*

$$\iint_D f(x, y)\, dx\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

A double integral on a Type II domain

Example

Find the integral of $f(x, y) = x^2 + y^2$ on the domain $D = \{(x, y) \in \mathbb{R}^2\ :\ y \leqslant x \leqslant \sqrt{y},\quad 0 \leqslant y \leqslant 1\}$.

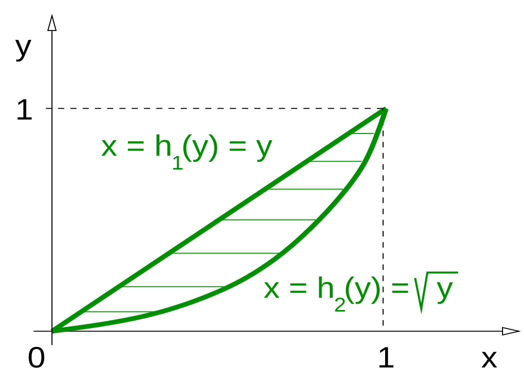
Solution:

This is a Type II domain, with left boundary

$$x = h_1(y) = y,$$

and right boundary

$$x = h_2(y) = \sqrt{y}.$$



Remark:

This domain is both Type I and Type II: $y = x^2 \Leftrightarrow x = \sqrt{y}$.

A double integral on a Type I domain

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \ : \ y \leqslant x \leqslant \sqrt{y}, \quad 0 \leqslant y \leqslant 1\}$.

Solution: $I = \iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$
with $h_1(y) = y$ and $h_2(y) = \sqrt{y}$, we obtain

$$I = \int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2)\,dx\,dy,$$

$$I = \int_0^1 \Big[\Big(\frac{x^3}{3}\Big|_y^{\sqrt{y}}\Big) + y^2\Big(x\Big|_y^{\sqrt{y}}\Big)\Big]\,dy,$$

$$I = \int_0^1 \Big[\frac{1}{3}\big(y^{3/2} - y^3\big) + y^2\big(y^{1/2} - y\big)\Big]\,dy.$$

A double integral on a Type I domain

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \ : \ y \leqslant x \leqslant \sqrt{y}, \quad 0 \leqslant y \leqslant 1\}$.

Solution: $I = \int_0^1 \Big[\frac{1}{3}\big(y^{3/2} - y^3\big) + y^2\big(y^{1/2} - y\big)\Big]\,dy.$

$$I = \int_0^1 \Big[\frac{1}{3}y^{3/2} - \frac{1}{3}y^3 + y^{5/2} - y^3\Big]\,dy,$$

$$I = \Big[\frac{1}{3}\,\frac{2}{5}\,y^{5/2} - \frac{1}{3}\,\frac{y^4}{4} + \frac{2}{7}\,y^{7/2} - \frac{y^4}{4}\Big]\Big|_0^1,$$

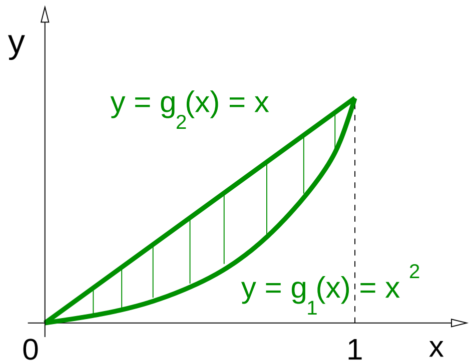
$$I = \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{(3)(5)(7)}.$$

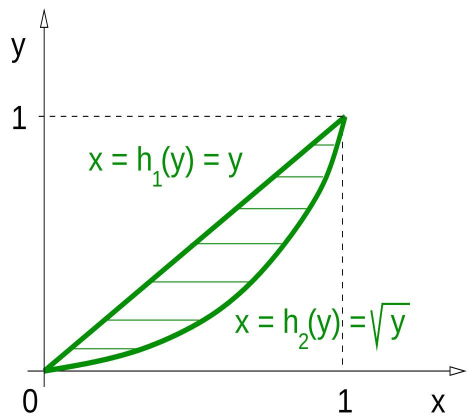
We conclude $\iint_D f(x,y)\,dx\,dy = \frac{3}{35}$. $\triangleleft$

Domains Type I and Type II

Summary: We have shown that a double integral of a function f on the domain D given in the pictures below holds,

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} f(x,y)\,dy\,dx = \int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy.$$





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- **Finding the limits of integration.**

Domains Type I and Type II

Example

Find the limits of integration of $\iint_D f(x,y)\,dxdy$ in the domain $D = \left\{(x,y) \in \mathbb{R}^2 \;:\; \frac{x^2}{9} + \frac{y^2}{4} \leqslant 1\right\}$ when D is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
So, the boundary as Type I is given by

$$y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}} = g_2(x).$$

The boundary as Type II is given by

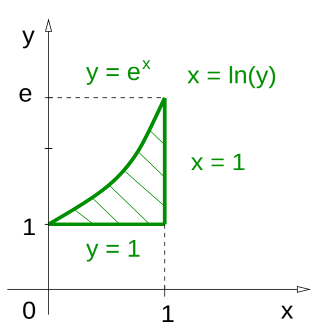
$$x = -3\sqrt{1 - \frac{y^2}{4}} = h_1(y), \quad x = 3\sqrt{1 - \frac{y^2}{4}} = h_2(y). \qquad \triangleleft$$

Domains Type I and Type II

Example

Reverse the order of integration in $\int_0^1 \int_1^{e^x} dy\,dx$.

Solution:
This integral is written as Type I, since we first integrate on vertical intervals $[1, e^x]$, with boundaries $y = e^x$, $y = 1$, while $x \in [0, 1]$.



Invert the first equation and from the figure we get the left and right boundaries:

$$x = \ln(y), \quad x = 1, \quad \text{with} \quad y \in [1, e].$$

Therefore, we conclude that $\int_0^1 \int_1^{e^x} dy\,dx = \int_1^e \int_{\ln(y)}^1 dx\,dy$. $\triangleleft$