Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- Double integral on rectangles.
- Fubini Theorem on rectangular domains.
- Examples.

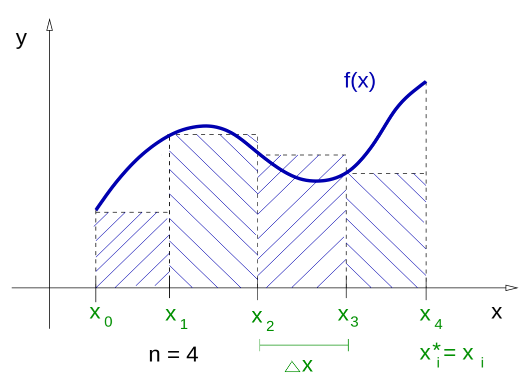
Review: Integral of a single variable function

Definition

The definite integral of a function $f : [a, b] \to \mathbb{R}$, in the interval $[a, b]$ is the number

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\, \Delta x.$$

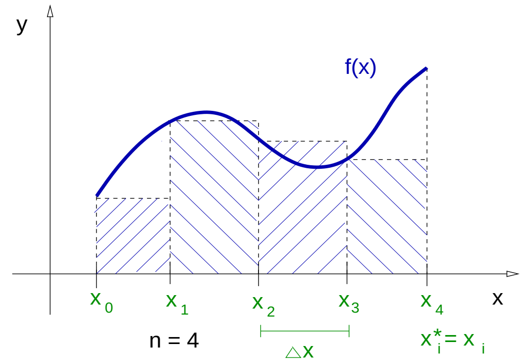
where $x_i^* \in [x_i, x_{i+1}]$ is called a sample point, while $\{x_i\}$ is a partition in $[a, b]$, $i = 0, \cdots, n$, and with $x_i = a + i\Delta x$, and $\Delta x = \dfrac{(b-a)}{n}$.



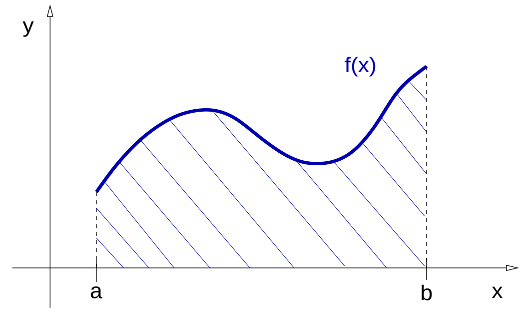
The integral as an area.

The sum $S_n = \sum_{i=0}^{n} f(x_i^*)\,\Delta x$ is called a Riemann sum. Then,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n.$$



The integral $\int_a^b f(x)\,dx$ is the area in between the graph of f and the horizontal axis.



Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- **Double integral on rectangles.**
- Fubini Theorem on rectangular domains.
- Examples.

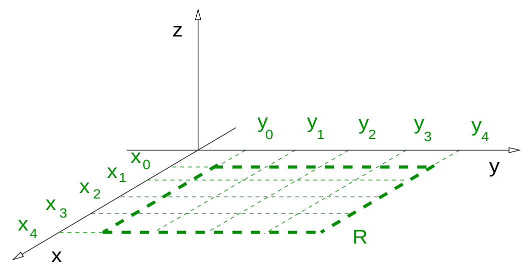
Double integrals on rectangles

Definition

The *double integral* of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ in the rectangle $R = [a, b] \times [c, d]$ is the number

$$\iint_R f(x, y)\, dx\, dy = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\, \Delta x \Delta y.$$

where $x_i^* \in [x_i, x_{i+1}]$, $y_j^* \in [y_j, y_{j+1}]$, are sample points, while $\{x_i\}$ and $\{y_j\}$, $i, j = 0, \cdots, n$ are partitions of the intervals $[a, b]$ and $[c, d]$, and $\Delta x = \dfrac{(b-a)}{n}$, $\Delta y = \dfrac{(d-c)}{n}$.

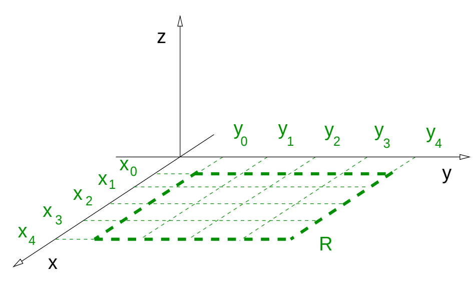


The double integral as a volume

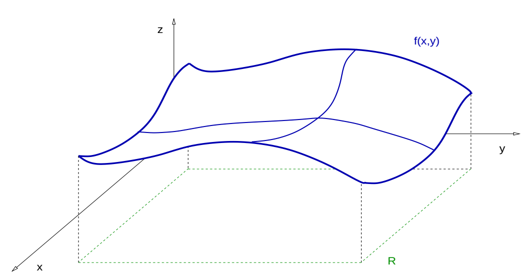
The sum

$S_n = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\, \Delta x \Delta y$ is called a Riemann sum. Then,

$$\iint_R f(x, y)\, dx\, dy = \lim_{n\to\infty} S_n.$$



The integral $\iint_R f(x, y)\, dx\, dy$ is the volume above R and below the graph of f.



Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- Double integral on rectangles.
- **Fubini Theorem on rectangular domains.**
- Examples.

Fubini Theorem on rectangular domains

Theorem

If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in $R = [x_0, x_1] \times [y_0, y_1]$, then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_0}^{y_1} \Big[\int_{x_0}^{x_1} f(x,y)\,dx\Big]\,dy,$$

$$= \int_{x_0}^{x_1} \Big[\int_{y_0}^{y_1} f(x,y)\,dy\Big]\,dx.$$

Remark: Fubini's Theorem: The order of integration can be switched in double integrals of continuous functions on a rectangle.

Notation: The double integral is also written as

$$\iint_R f(x,y)\,dx\,dy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x,y)\,dx\,dy.$$

Fubini Theorem on rectangular domains

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: Since $x \in [0,2]$ and $y \in [1,3]$,

$$I = \iint_R f(x,y)\,dx\,dy = \int_1^3 \int_0^2 (xy^2 + 2x^2y^3)dx\,dy$$

$$I = \int_1^3 \Big[\int_0^2 (xy^2 + 2x^2y^3)dx\Big]\,dy.$$

We compute the interior integral in x first, keeping y constant. After that we compute the integral in y.

Fubini Theorem on rectangular domains

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: We compute the integral in x first, keeping y constant.

$$I = \iint_R f(x,y)\,dx\,dy = \int_1^3 \Big[\int_0^2 (xy^2 + 2x^2y^3)dx\Big]\,dy,$$

$$I = \int_1^3 \Big[\frac{y^2}{2}\Big(x^2\Big|_0^2\Big) + \frac{2y^3}{3}\Big(x^3\Big|_0^2\Big)\Big]\,dy,$$

$$I = \int_1^3 \Big[2y^2 + \frac{16}{3}y^3\Big]\,dy.$$

We now compute the integral in y.

Fubini Theorem on rectangular domains

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: We now compute the integral in y.

$$I = \int_1^3 \Big[2y^2 + \frac{16}{3}y^3\Big]\,dy = 2\,\frac{y^3}{3}\Big|_1^3 + \frac{16}{3}\,\frac{y^4}{4}\Big|_1^3.$$

$$I = 2\,\frac{26}{3} + \frac{4}{3}80 = \frac{372}{3}.$$

We conclude: $\iint_R f(x,y)\,dx\,dy = \frac{372}{3}.$ $\triangleleft$

Fubini Theorem on rectangular domains

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in y, then in x.

Solution:

$$I = \iint_R f(x,y)\,dx\,dy = \int_1^3 \int_0^2 (xy^2 + 2x^2y^3)dx\,dy$$

$$I = \int_0^2 \Big[\int_1^3 (xy^2 + 2x^2y^3)dy\Big]\,dx.$$

$$I = \int_0^2 \Big[\frac{x}{3}\Big(y^3\Big|_1^3\Big) + \frac{2x^2}{4}\Big(y^4\Big|_1^3\Big)\Big]\,dx.$$

Fubini Theorem on rectangular domains

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: $I = \int_0^2 \Big[\frac{x}{3}\Big(y^3\Big|_1^3\Big) + \frac{2x^2}{4}\Big(y^4\Big|_1^3\Big)\Big]\,dx.$

$$I = \int_0^2 \Big[\frac{26}{3}\,x + 40\,x^2\Big]\,dx = \frac{26}{3}\,\frac{x^2}{2}\Big|_0^2 + 40\,\frac{x^3}{3}\Big|_0^2,$$

$$I = \frac{26}{3}\,(2) + 40\,\frac{8}{3} = \frac{372}{3}.$$

We conclude: $\iint_R f(x,y)\,dx\,dy = \frac{372}{3}.$ ◁

Fubini Theorem on rectangular domains

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = \frac{x}{y} + \frac{y}{x}$, and $R = [1,4] \times [1,2]$.

Solution: We choose to first integrate in y and then in x.

$$I = \iint_R f(x,y)\,dx\,dy = \int_1^4 \int_1^2 \Big(\frac{x}{y} + \frac{y}{x}\Big)\,dy\,dx,$$

$$I = \int_1^4 \Big[\int_1^2 \Big(\frac{x}{y} + \frac{y}{x}\Big)dy\Big]\,dx = \int_1^4 \Big[x\Big(\ln(y)\Big|_1^2\Big) + \frac{1}{x}\Big(\frac{y^2}{2}\Big|_1^2\Big)\Big]\,dx$$

$$I = \int_1^4 \Big[\ln(2)\,x + \frac{3}{2}\,\frac{1}{x}\Big]\,dx.$$

Fubini Theorem on rectangular domains

Example

Use Fubini's Theorem to compute the double integral
$\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$, and $R = [1,4] \times [1,2]$.

Solution: We compute the integral in x,

$$I = \int_1^4 \Big[\ln(2)\,x + \frac{3}{2}\,\frac{1}{x}\Big]\,dx = \ln(2)\Big(\frac{x^2}{2}\Big|_1^4\Big) + \frac{3}{2}\Big(\ln(x)\Big|_1^4\Big),$$

$$I = \frac{15}{2}\ln(2) + \frac{3}{2}\ln(4) = \Big(\frac{15}{2} + 3\Big)\ln(2).$$

We conclude: $\iint_R f(x,y)\,dx\,dy = \dfrac{21}{2}\,\ln(2).$ ◁

A particular case of Fubini's Theorem

Corollary

If the continuous function $f : R \subset R^2 \to \mathbb{R}$ satisfies that $f(x,y) = g(x)h(y)$, then the double integral of function f in the rectangle $R = [x_0, x_1] \times [y_0, y_1]$ is given by

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} g(x)h(y)dy\,dx = \Big(\int_{x_0}^{x_1} g(x)dx\Big)\Big(\int_{y_0}^{y_1} h(y)dy\Big).$$

Remark: In the case that $f(x,y)$ is a product of two functions g, h, with $g(x)$ and $h(y)$, then the double integral of f is also a product of the integral of g times the integral of h.

A particular case of Fubini's Theorem

Example

Compute the double integral of $f(x,y) = \dfrac{1+x^2}{1+y^2}$, in the rectangular region $R = [0,2] \times [0,1]$.

Solution: $I = \displaystyle\iint_R f(x,y)\,dx\,dy = \int_0^2 \int_0^1 \frac{1+x^2}{1+y^2} dy\,dx,$

$$I = \Big[\int_0^2 (1+x^2)dx\Big]\Big[\int_0^1 \frac{1}{1+y^2} dy\Big],$$

$$I = \Big(x\Big|_0^2 + \frac{1}{3}x^3\Big|_0^2\Big)\Big(\arctan(y)\Big|_0^1\Big) = \Big(2 + \frac{8}{3}\Big)\frac{\pi}{4} = \frac{14}{3}\frac{\pi}{4}$$

We conclude $\displaystyle\iint_R f(x,y)\,dx\,dy = \frac{7}{6}\pi.$ $\triangleleft$