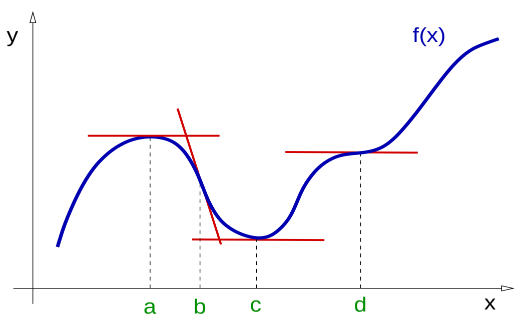
Local and absolute extrema, saddle points (Sect. 14.7)

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.

Review: Local extrema for functions of one variable

Recall: Main results on local extrema for $f(x)$:



at	f	f'	f''
a	max.	0	< 0
b	infl.	$\neq 0$	$\pm\ 0\ \mp$
c	min.	0	> 0
d	infl.	$= 0$	$\pm\ 0\ \mp$

Remarks: Assume that f is twice continuously differentiable.

- If x_0 is local maximum or minimum of f, then $f'(x_0) = 0$.
- If $f'(x_0) = 0$, then x_0 is a critical point of f, that is, x_0 is a maximum or a minimum or an inflection point.
- The second derivative test determines whether a critical point is a maximum, minimum or an inflection point.

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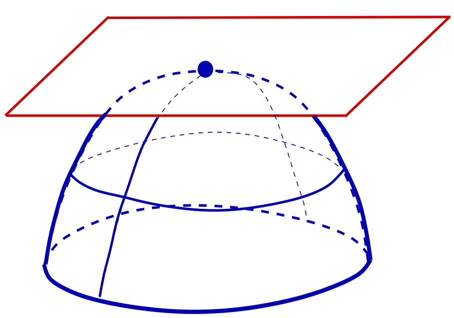
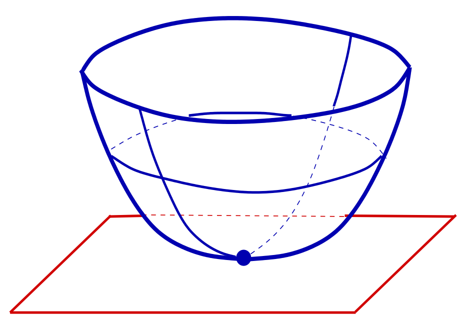
Definition of local extrema for functions of two variables

Definition

A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *local maximum* at the point $(a, b) \in D$ iff holds that $f(a, b) \geqslant f(x, y)$ for every point (x, y) in a neighborhood of (a, b).
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *local minimum* at the point $(a, b) \in D$ iff holds that $f(a, b) \leqslant f(x, y)$ for every point (x, y) in a neighborhood of (a, b).

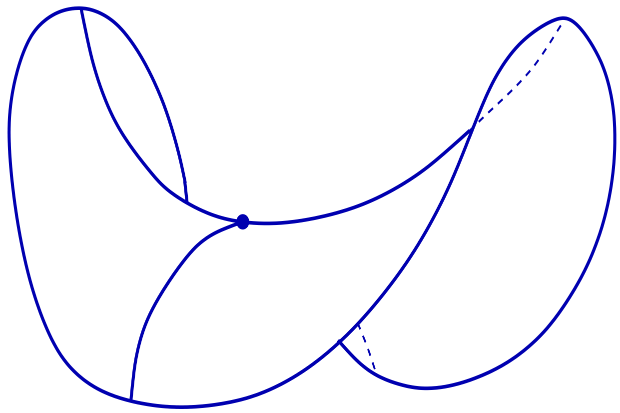




Definition of local extrema for functions of two variables

Definition

A differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *saddle point* at an interior point $(a, b) \in D$ iff in every open disk in D centered at (a, b) there always exist points (x, y) where $f(a, b) < f(x, y)$ and other points (x, y) where $f(a, b) > f(x, y)$.



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Characterization of local extrema

Theorem (First Derivative Test)

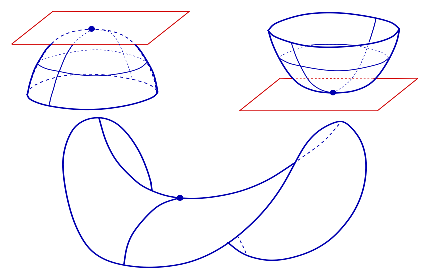
If a differentiable function f has a local maximum or minimum at (a,b) then holds $\big(\nabla f\big)\big|_{(a,b)} = \langle 0,0\rangle$.

Remark: The tangent plane at a local extremum is horizontal, since its normal vector is $\mathbf{n} = \langle f_x, f_y, -1\rangle = \langle 0,0,-1\rangle$.

Definition

The interior point $(a,b) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a *critical point* of f iff $\big(\nabla f\big)\big|_{(a,b)} = \langle 0,0\rangle$.

Remark:
Critical points include local maxima, local minima, and saddle points.



Characterization of local extrema

Example

Find the critical points of the function $f(x,y) = -x^2 - y^2$.

Solution: The critical points are the points where ∇f vanishes. Since $\nabla f = \langle -2x, -2y\rangle$, the only solution to $\nabla f = \langle 0,0\rangle$ is $x = 0$, $y = 0$. That is, $(a,b) = (0,0)$. $\triangleleft$

Remark: Since $f(x,y) \leqslant 0$ for all $(x,y) \in \mathbb{R}^2$ and $f(0,0) = 0$, then the point $(0,0)$ must be a local maximum of f.

Example

Find the critical points of the function $f(x,y) = x^2 - y^2$.

Solution: Since $\nabla f = \langle 2x, -2y\rangle$, the only solution to $\nabla f = \langle 0,0\rangle$ is $x = 0$, $y = 0$. That is, we again obtain $(a,b) = (0,0)$. $\triangleleft$

Characterization of local extrema

Theorem (Second derivative test)

Let (a,b) be a critical point of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, that is, $(\nabla f)\big|_{(a,b)} = \langle 0, 0\rangle$. Assume that f has continuous second derivatives in an open disk in D with center in (a,b) and denote

$$D = f_{xx}(a,b)\, f_{yy}(a,b) - \big[f_{xy}(a,b)\big]^2.$$

Then, the following statements hold:

- *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*
- *If $D < 0$, then $f(a,b)$ is a saddle point.*
- *If $D = 0$ the test is inconclusive.*

Notation: The number D is called the *discriminant* of f at (a,b).

Characterization of local extrema

Example

Find the local extrema of $f(x,y) = y^2 - x^2$ and determine whether they are local maximum, minimum, or saddle points.

Solution: We first find the critical points:

$$\nabla f = \langle -2x, 2y\rangle \quad \Rightarrow \quad (\nabla f)\big|_{(a,b)} = \langle 0,0\rangle \;\text{ iff }\; (a,b) = (0,0).$$

The only critical point is $(a,b) = (0,0)$.

We need to compute $D = f_{xx}(a,b)\, f_{yy}(a,b) - \big[f_{xy}(a,b)\big]^2$.

Since $f_{xx}(0,0) = -2$, $f_{yy}(0,0) = 2$, and $f_{xy}(0,0) = 0$, we get

$$D = (-2)(2) = -4 < 0 \quad \Rightarrow \quad \text{saddle point at } (0,0). \qquad \triangleleft$$

Characterization of local extrema.

Example

Is the point $(a, b) = (0, 0)$ a local extrema of $f(x, y) = y^2x^2$?

Solution: We first verify that $(0, 0)$ is a critical point of f:

$$\nabla f(x, y) = \langle 2xy^2, 2yx^2 \rangle, \quad \Rightarrow \quad (\nabla f)\big|_{(0,0)} = \langle 0, 0 \rangle,$$

therefore, $(0, 0)$ is a critical point.

Remark: The whole axes $x = 0$ and $y = 0$ are critical points of f.

We need to compute $D = f_{xx}(a, b)\, f_{yy}(a, b) - \big[f_{xy}(a, b)\big]^2$.

Since $f_{xx}(x, y) = 2y^2$, $f_{yy}(x, y) = 2x^2$, and $f_{xy}(x, y) = 4xy$,

we obtain $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$, and $f_{xy}(0, 0) = 0$,

hence $D = 0$ and the test is inconclusive. $\triangleleft$

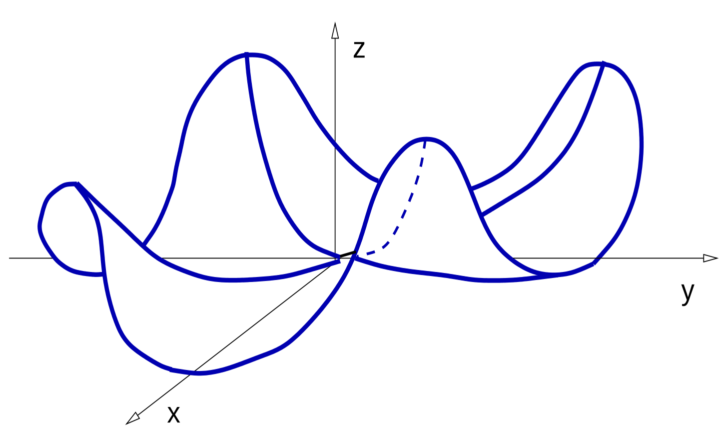
Characterization of local extrema.

Example

Is the point $(a, b) = (0, 0)$ a local extrema of $f(x, y) = y^2x^2$?

Solution: Since $f(x, y) = x^2y^2 \geqslant 0$ for all (x, y), and $f(0, 0) = 0$, then $(0, 0)$ is a local minimum. (Also a global minimum.) $\triangleleft$

This is confirmed in the graph of f.



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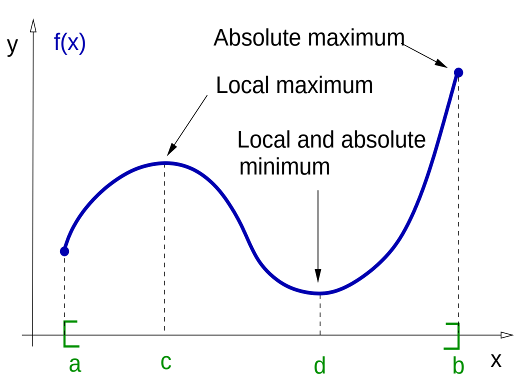
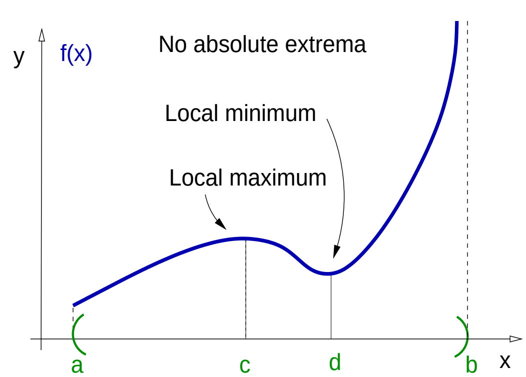
Absolute extrema of a function in a domain

Definition
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute maximum at the point $(a, b) \in D$ iff $f(a, b) \geqslant f(x, y)$ for all $(x, y) \in D$.
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute minimum at the point $(a, b) \in D$ iff $f(a, b) \leqslant f(x, y)$ for all $(x, y) \in D$.

Remark: Local extrema need not be the absolute extrema.



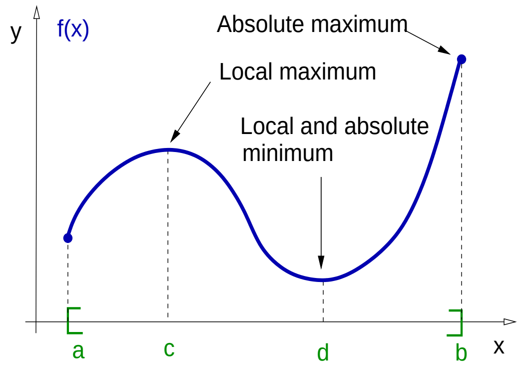
Remark: Absolute extrema may not be defined on open intervals.



Review: Functions of one variable

Theorem

Every continuous function on a closed interval, $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, with $a < b \in \mathbb{R}$, always has absolute extrema.



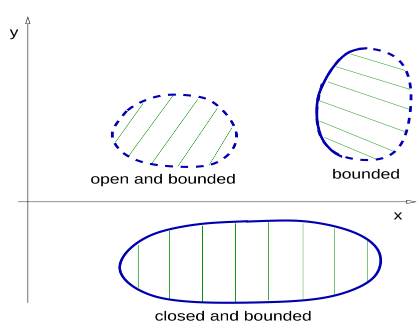
Recall:

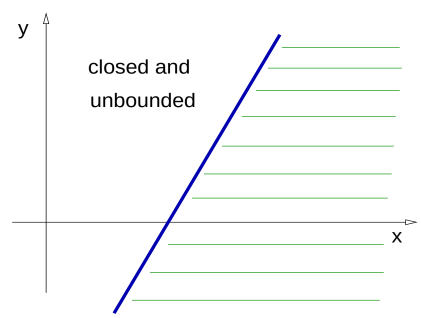
- Intervals $[a, b]$ are bounded and closed sets in $\mathbb{R}$.
- The set $[a, b]$ is closed, since the boundary points belong to the set, and it is bounded, since it does not extend to infinity.

Recall: On open and closed sets in $\mathbb{R}^n$

Definition

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in S is an interior point. The set S is called *closed* iff S contains its boundary. A set S is called *bounded* iff S is contained in ball, otherwise S is called *unbounded*.





Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in a closed and bounded set D, then f has an absolute maximum and an absolute minimum in D.

Absolute extrema on closed and bounded sets

Problem:

Find the absolute extrema of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ in a closed and bounded set D.

Solution:

(1) Find every critical point of f in the interior of D and evaluate f at these points.

(2) Find the boundary points of D where f has local extrema, and evaluate f at these points.

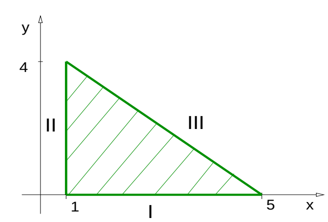
(3) Look at the list of values for f found in the previous two steps.

If $f(x_0, y_0)$ is the biggest (smallest) value of f in the list above, then (x_0, y_0) is the absolute maximum (minimum) of f in D.

Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



Solution:

(1) We find all critical points in the interior of the domain:

$$\nabla f = \langle (y-1), (x+2) \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad (x_0, y_0) = (-2, 1).$$

Since $(-2, 1)$ does not belong to the domain, we discard it.

(2) Three segments form the boundary of D:

Boundary I: The segment $y = 0$, $x \in [1, 5]$. We select the end points $(1, 0)$, $(5, 0)$, and we record: $f(1, 0) = 2$ and $f(5, 0) = -2$.

We look for critical point on the interior of Boundary I:

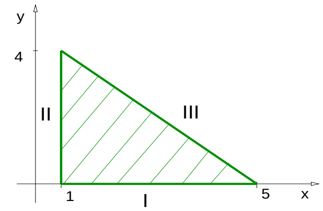
Since $g(x) = f(x, 0) = 3 - x$, so $g' = -1 \neq 0$.

No critical points in the interior of Boundary I.

Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$. We select the end point $(1, 4)$ and we record: $f(1, 4) = 14$.

We look for critical point on the interior of Boundary II:

Since $g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y$, so $g' = 3 \neq 0$.

No critical points in the interior of Boundary II.

Boundary III: The segment $y = -x + 5$, $x \in [1, 5]$.

We look for critical point on the interior of Boundary III:

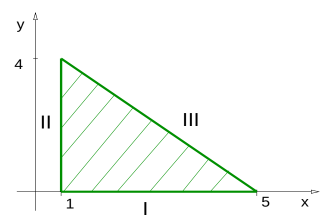
Since $g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5)$.

We obtain $g(x) = -x^2 + 2x + 13$, hence $g'(x) = -2x + 2 = 0$ implies $x = 1$. So, $y = 4$, and we selected the point $(1, 4)$, which was already in our list. No critical points in the interior of III.

Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



Solution:

(3) Our list of values is:

$$f(1, 0) = 2 \quad f(1, 4) = 14 \quad f(5, 0) = -2.$$

We conclude:

(a) Absolute maximum at $(1, 4)$,

(b) Absolute minimum at $(5, 0)$.

◁

A maximization problem with a constraint

Example

Find the maximum volume of a closed rectangular box with a given surface area A_0.

Solution: This problem can be solved by finding the local maximum of an appropriate function f.

First, the functions volume and area of a rectangular box with vertex at $(0,0,0)$ and sides x, y and z are:

$$V(x,y,z) = xyz, \quad A(x,y,z) = 2xy + 2xz + 2yz.$$

Since $A(x,y,z) = A_0$, we obtain

$$z = \frac{A_0 - 2xy}{2(x+y)} \quad \Rightarrow \quad f(x,y) = \frac{A_0 xy - 2x^2y^2}{2(x+y)}.$$

A maximization problem with a constraint

Example

Find the maximum volume of a closed rectangular box with a given surface area A_0.

Solution: Find the critical points of $f(x,y) = \dfrac{A_0 xy - 2x^2y^2}{2(x+y)}$.

$$f_x = \frac{2A_0 y^2 - 4x^2y^2 - 8xy^3}{4(x+y)^2}, \quad f_y = \frac{2A_0 x^2 - 4x^2y^2 - 8yx^3}{4(x+y)^2}.$$

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0$, $y \neq 0$ imply

$$\left.\begin{aligned} A_0 &= 2x^2 + 4xy, \\ A_0 &= 2y^2 + 4xy, \end{aligned}\right\} \Rightarrow \; x = y \; \Rightarrow \; A_0 = 2x^2 + 4x^2.$$

Then, $x_0 = \sqrt{\dfrac{A_0}{6}} = y_0$. Since $z = \dfrac{A_0 - 2xy}{2(x+y)}$, $z_0 = \sqrt{\dfrac{A_0}{6}}$. ◁