Review for Exam 2

- Sections 13.1-13.3. 14.1-14.7.
- 50 minutes.
- 5, 6 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Section 14.7

Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle$, then,

$$x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x - 4) = 0.$$

There are two solutions, $x = 0 \Rightarrow y = 0$, and $x = 4 \Rightarrow y = 8$.

That is, there are two critical points, $(0, 0)$ and $(4, 8)$.

Section 14.7

Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:

(b) Recalling $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle$, we compute

$$f_{xx} = -12x, \quad f_{yy} = -6, \quad f_{xy} = 12.$$

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144 \left(\frac{x}{2} - 1 \right),$$

Since $D(0, 0) = -144 < 0$, the point $(0, 0)$ is a saddle point of f.

Since $D(4, 8) = 144(2 - 1) > 0$, and $f_{xx}(4, 8) = (-12)4 < 0$, the point $(4, 8)$ is a local maximum of f. $\triangleleft$

Section 14.6

Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution:

(a) $L(x, y) = f_x(-3, 2)\,(x + 3) + f_y(-3, 2)\,(y - 2) + f(-3, 2)$.

Since $f_x(x, y) = 2\cos(2x + 3y)$ and $f_y(x, y) = 3\cos(2x + 3y)$,

$$f_x(-3, 2) = 2\cos(-6 + 6) = 2, \quad f_y(-3, 2) = 3\cos(-6 + 6) = 3,$$

$$f(-3, 2) = \sin(-6 + 6) + 1 = 1.$$

the linear approximation is $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

Section 14.6

Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution: Recall: $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

(b) We use L to find the a linear approximation to $f(-2.9, 2.1)$.

We need to compute $L(-2.9, 2.1)$.

$$L(-2.9, 2.1) = 2(-2.9 + 3) + 3(2.1 - 2) + 2$$

$$L(-2.9, 2.1) = 2(0.1) + 3(0.1) + 1 \quad \Rightarrow \quad L(-2.9, 2.1) = 1.5.$$

$\triangleleft$

Exact value is close to 1.479.

Section 14.5

Example

(a) Find the gradient of $f(x, y, z) = \sqrt{x + 2yz}$.

(b) Find the directional derivative of f at $(0, 2, 1)$ in the direction given by $\langle 0, 3, 4 \rangle$.

(c) Find the maximum rate of change of f at the point $(0, 2, 1)$.

Solution:

(a) $\nabla f(x, y, z) = \dfrac{1}{2\sqrt{x + 2yz}} \langle 1, 2z, 2y \rangle$.

(b) We evaluate the gradient above at $(0, 2, 1)$,

$$\nabla f(0, 2, 1) = \frac{1}{2\sqrt{0 + 4}} \langle 1, 2, 4 \rangle = \frac{1}{4} \langle 1, 2, 4 \rangle.$$

Section 14.5

Example

(a) Find the gradient of $f(x, y, z) = \sqrt{x + 2yz}$.

(b) Find the directional derivative of f at $(0, 2, 1)$ in the direction given by $\langle 0, 3, 4\rangle$.

(c) Find the maximum rate of change of f at the point $(0, 2, 1)$.

Solution: (b) Recall: $\nabla f(0, 2, 1) = \frac{1}{4}\langle 1, 2, 4\rangle$.

We now need a unit vector parallel to $\langle 0, 3, 4\rangle$,

$$\mathbf{u} = \frac{1}{\sqrt{9+16}}\langle 0, 3, 4\rangle = \frac{1}{5}\langle 0, 3, 4\rangle.$$

Then, $(D_u f)(0, 2, 1) = \frac{1}{4}\langle 1, 2, 4\rangle \cdot \frac{1}{5}\langle 0, 3, 4\rangle = \frac{1}{20}(6 + 16) = \frac{11}{10}$.

We obtain, $(D_u f)(0, 2, 1) = \frac{11}{10}$.

Section 14.5

Example

(a) Find the gradient of $f(x, y, z) = \sqrt{x + 2yz}$.

(b) Find the directional derivative of f at $(0, 2, 1)$ in the direction given by $\langle 0, 3, 4\rangle$.

(c) Find the maximum rate of change of f at the point $(0, 2, 1)$.

Solution:

(c) The maximum rate of change of f at a point is the magnitude of its gradient at that point, that is,

$$|\nabla f(0, 2, 1)| = \frac{1}{4}|\langle 1, 2, 4\rangle| = \frac{1}{4}\sqrt{1 + 4 + 16} = \frac{\sqrt{21}}{4}.$$

The maximum rate of change of f at $(0, 2, 1)$ is

$$|\nabla f(0, 2, 1)| = \sqrt{21}/4.$$

$\triangleleft$

Section 14.4

Example

Find $\partial_{xy}\big(e^{-xy}\sin(x+yz)\big)$. (Do not simplify your answer.)

Solution: We first compute the x-derivative,

$$\partial_x\big(e^{-xy}\sin(x+yz)\big) = -ye^{-xy}\sin(x+yz) + e^{-xy}\cos(x+yz).$$

The second derivative is

$$\partial_{xy}\big(e^{-xy}\sin(x+yz)\big) = \partial_y\Big(-ye^{-xy}\sin(x+yz)+e^{-xy}\cos(x+yz)\Big),$$

$$= -e^{-xy}\sin(x+yz) + xye^{-xy}\sin(x+yz) - ye^{-xy}\cos(x+yz)z$$

$$-xe^{-xy}\cos(x+yz) - e^{-xy}\sin(x+yz)z. \qquad \triangleleft$$

Section 14.3

Example

Find any value of the constant a such that the function $f(x,y) = e^{-ax}\cos(y) - e^{-y}\cos(x)$ is solution of Laplace's equation $f_{xx} + f_{yy} = 0$.

Solution:

$$f_x = -ae^{-ax}\cos(y) + e^{-y}\sin(x), \quad f_{xx} = a^2e^{-ax}\cos(y) + e^{-y}\cos(x).$$

$$f_y = -e^{-ax}\sin(y) + e^{-y}\cos(x), \quad f_{yy} = -e^{-ax}\cos(y) - e^{-y}\cos(x).$$

$$f_{xx} + f_{yy} = \big[a^2e^{-ax}\cos(y) + e^{-y}\cos(x)\big] + \big[-e^{-ax}\cos(y) - e^{-y}\cos(x)\big],$$

$$f_{xx} + f_{yy} = (a^2-1)e^{-ax}\cos(y).$$

Function f is solution of $f_{xx} + f_{yy} = 0$ iff $a = \pm 1$. $\triangleleft$

Section 14.2

Example

Compute the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2 \sin^2(y)}{2x^2 + 3y^2}$.

Solution:

Since $x^2 \leqslant 2x^2 + 3y^2$, that is, $\dfrac{x^2}{2x^2+3y^2} \leqslant 1$, the non-negative function $f(x,y) = \dfrac{x^2 \sin^2(y)}{2x^2+3y^2}$ satisfies the bounds

$$0 \leqslant f(x,y) \leqslant \sin^2(y).$$

Since $\lim_{y\to 0} \sin^2(y) = 0$, the Sandwich Theorem implies that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2(y)}{2x^2+3y^2} = 0. \qquad \triangleleft$$

Section 13.3

Example

Reparametrize the curve $\mathbf{r}(t) = \left\langle \frac{3}{2}\sin(t^2), 2t^2, \frac{3}{2}\cos(t^2) \right\rangle$ with respect to its arc length measured from $t = 1$ in the direction of increasing t.

Solution:

We first compute the arc length function. We start with the derivative

$$\mathbf{r}'(t) = \langle 3t\cos(t^2), 4t, -3\sin(t^2)\rangle,$$

We now need its magnitude,

$$|\mathbf{r}'(t)| = \sqrt{9t^2\cos^2(t^2) + 16t^2 + 9\sin^2(t^2)},$$

$$|\mathbf{r}'(t)| = \sqrt{9t^2 + 16t^2} = \left(\sqrt{9+16}\right) t \quad \Rightarrow \quad |\mathbf{r}'(t)| = 5t.$$

Section 13.3

Example

Reparametrize the curve $\mathbf{r}(t) = \left\langle \frac{3}{2}\sin(t^2), 2t^2, \frac{3}{2}\cos(t^2) \right\rangle$ with respect to its arc length measured from $t = 1$ in the direction of increasing t.

Solution: Recall: $|\mathbf{r}'(t)| = 5t$. The arc length function is

$$s(t) = \int_1^t 5\tau\, d\tau = \frac{5}{2}\Big(\tau^2\Big|_1^t\Big) = \frac{5}{2}(t^2 - 1).$$

Inverting this function for t^2, we obtain $t^2 = \frac{2}{5}s + 1$.
The reparametrization of $\mathbf{r}(t)$ is given by

$$\hat{\mathbf{r}}(s) = \left\langle \frac{3}{2}\sin\Big(\frac{2}{5}s + 1\Big), 2\Big(\frac{2}{5}s + 1\Big), \frac{3}{2}\cos\Big(\frac{2}{5}s + 1\Big) \right\rangle. \quad \triangleleft$$