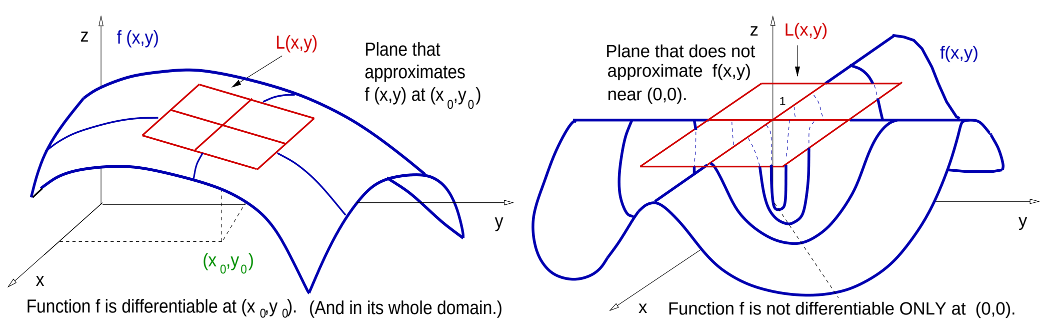
## Tangent planes and linear approximations (Sect. 14.6)

- Review: Differentiable functions of two variables.
- The tangent plane to the graph of a function.
- The linear approximation of a differentiable function.
- Bounds for the error of a linear approximation.
- The differential of a function.
  - Review: Scalar functions of one variable.
  - Scalar functions of more than one variable.

## Review: Differentiable functions of two variables.

Recall: The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in $D$.




$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

**Theorem**

*If the partial derivatives $f_x$ and $f_y$ of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then $f$ is differentiable in $R$.*

## Review: Differentiable functions of two variables

Example

Show that the function $f(x, y) = x^2 + y^2$ is differentiable for all $(x, y) \in \mathbb{R}^2$. Furthermore, find the linear function $L$, mentioned in the definition of a differentiable function, at the point $(1, 2)$.

Solution: We need to compute the partial derivatives of $f$.
$f_x(x, y) = 2x$ and $f_y(x, y) = 2y$. They are continuous functions, then $f$ is differentiable. The linear function $L$ at $(1, 2)$ is

$$L(x, y) = f_x(1, 2)\,(x - 1) + f_y(1, 2)\,(y - 2) + f(1, 2).$$

That is, we need three numbers to find the linear function $L$: $f_x(1, 2)$, $f_y(1, 2)$, and $f(1, 2)$. These numbers are:

$$f_x(1, 2) = 2, \quad f_y(1, 2) = 4, \quad f(1, 2) = 5.$$

Therefore, $L(x, y) = 2(x - 1) + 4(y - 2) + 5$. $\triangleleft$

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## The tangent plane to the graph of a function

Remark:

The function $L(x,y) = 2(x-1) + 4(y-2) + 5$ is a plane in $\mathbb{R}^3$.
We usually write down the equation of a plane using the notation $z = L(x,y)$, that is, $z = 2(x-1) + 4(y-2) + 5$, or equivalently

$$2(x-1) + 4(y-2) - (z-5) = 0.$$

This is a plane passing through $\tilde{P}_0 = (1,2,5)$ with normal vector $\mathbf{n} = \langle 2, 4, -1 \rangle$. Analogously, the function

$$L(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$

is a plane in $\mathbb{R}^3$. Using the notation $z = L(x,y)$ we obtain

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0.$$

This is a plane passing through $\tilde{P}_0 = (x_0, y_0, f(x_0,y_0))$ with normal vector $\mathbf{n} = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$.

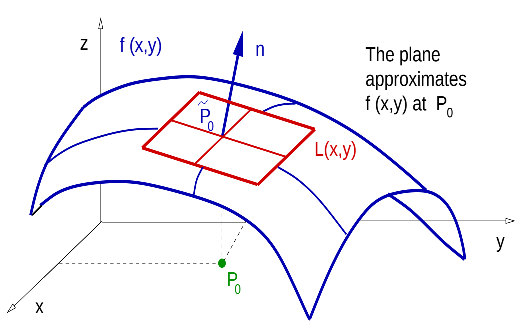
## The tangent plane to the graph of a function

Theorem

*The plane tangent to the graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $(x_0, y_0)$ is given by*

$$L(x,y) = f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0) + f(x_0,y_0).$$

Proof



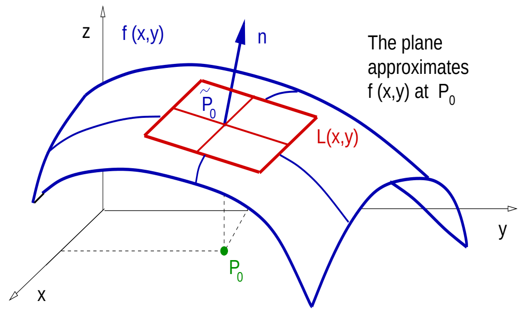

Since at $(x_0, y_0)$ the function $L$ satisfies that

$$L(x_0,y_0) = f(x_0,y_0).$$

then the plane contains the point $\tilde{P}_0 = (x_0, y_0, f(x_0,y_0))$.
We only need to find its normal vector $\mathbf{n}$.

# The tangent plane to the graph of a function.

The vector $\mathbf{n}$ normal to the plane $L(x, y)$ is a vector perpendicular to the surface $z = f(x, y)$ at $P_0 = (x_0, y_0)$.




This surface is the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = f(x, y) - z$. A vector normal to this level surface is its gradient $\nabla F$. That is, $\nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle$.

Therefore, the normal to the tangent plane $L(x, y)$ at the point $P_0$ is $\mathbf{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$. Recall that the plane contains the point $\tilde{P}_0 = (x_0, y_0, f(x_0, y_0))$. The equation for the plane is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0. \quad \square$$

# The tangent plane to the graph of a function.

Example

Show that $f(x, y) = \arctan(x + 2y)$ is differentiable and find the plane tangent to $f(x, y)$ at $(1, 0)$.

Solution: The partial derivatives of $f$ are given by

$$f_x(x, y) = \frac{1}{1 + (x + 2y)^2}, \quad f_y(x, y) = \frac{2}{1 + (x + 2y)^2}.$$

These functions are continuous in $\mathbb{R}^2$, so $f(x, y)$ is differentiable at every point in $\mathbb{R}^2$. The plane $L(x, y)$ at $(1, 0)$ is given by

$$L(x, y) = f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) + f(1, 0),$$

where $f(1, 0) = \arctan(1) = \pi/4$, $f_x(1, 0) = 1/2$, $f_y(1, 0) = 1$.

Then, $L(x, y) = \frac{1}{2}(x - 1) + y + \frac{\pi}{4}$. $\triangleleft$

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## The linear approximation of a differentiable function

**Definition**

The *linear approximation* of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $(x_0, y_0) \in D$ is the plane

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

**Example**

Find the linear approximation of $f = \sqrt{17 - x^2 - 4y^2}$ at $(2, 1)$.

Solution: $L(x, y) = f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) + f(2, 1)$.

We need three numbers: $f(2, 1)$, $f_x(2, 1)$, and $f_y(2, 1)$.

These are: $f(2, 1) = 3$, $f_x(2, 1) = -2/3$, and $f_y(2, 1) = -4/3$.

Then the plane is given by $L(x, y) = -\frac{2}{3}(x - 2) - \frac{4}{3}(y - 1) + 3.\lhd$

## Tangent planes and linear approximations (Sect. 14.6)

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## Bounds for the error of a linear approximation

### Theorem

*Assume that the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has first and second partial derivatives continuous on an open set containing a rectangular region $R \subset D$ centered at the point $(x_0, y_0)$.*
*If $M \in \mathbb{R}$ is the upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ in $R$, then the error $E(x,y) = f(x,y) - L(x,y)$ satisfies the inequality*

$$|E(x,y)| \leqslant \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2,$$

*where $L(x,y)$ is the linearization of $f$ at $(x_0, y_0)$, that is,*

$$L(x,y) = f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0) + f(x_0,y_0).$$

## Bounds for the error of a linear approximation

**Example**

Find an upper bound for the error in the linear approximation of $f(x,y) = x^2 + y^2$ at the point $(1,2)$ over the rectangle

$$R = \{(x,y) \in \mathbb{R}^2 \ : \ |x-1| < 0.1, \quad |y-2| < 0.1\}$$

Solution: The second derivatives of $f$ are $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$. Therefore, we can take $M = 2$.

Then the formula $|E(x,y)| \leqslant \dfrac{1}{2} M \big(|x-x_0| + |y-y_0|\big)^2$, implies

$$|E(x,y)| \leqslant \big(|x-1| + |y-2|\big)^2 < (0.1 + 0.1)^2 = 0.04,$$

that is $|E(x,y)| < 0.04$. $\triangleleft$

Since $f(1,2) = 5$, the % relative error is $100\,\dfrac{E(x,y)}{f(1,2)} \leqslant 0.8\%$.

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## Review: Differential of functions of one variable.

**Definition**

The *differential at* $x_0 \in D$ of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is the linear function

$$df(x) = L(x) - f(x_0).$$

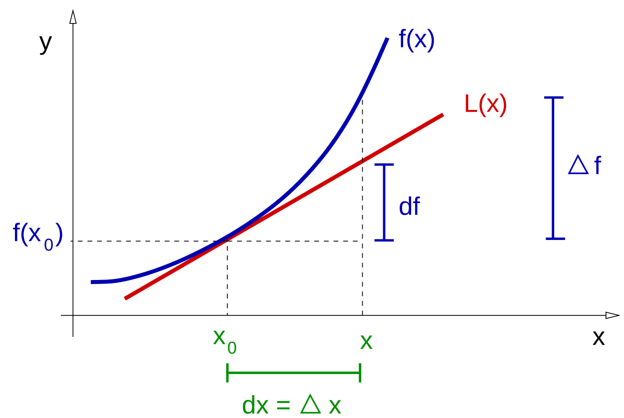
**Remark:** The linear approximation of $f(x)$ at $x_0$ is the line given by $L(x) = f'(x_0)\,(x - x_0) + f(x_0)$.

Therefore

$$df(x) = f'(x_0)\,(x - x_0).$$

Denoting $dx = x - x_0$,

$$df = f'(x_0)\,dx.$$




## Differential of functions of more than one variable

**Definition**

The *differential at* $(x_0, y_0) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the linear function

$$df(x, y) = L(x, y) - f(x_0, y_0).$$

**Remark:** The linear approximation of $f(x, y)$ at $(x_0, y_0)$ is the plane $L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0)$.

Therefore $df(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$.

Denoting $dx = x - x_0$ and $dy = (y - y_0)$ we obtain the usual expression

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy.$$

Therefore, $df$ and $L$ are similar concepts: The linear approximation of a differentiable function $f$.

## Differential of functions of more than one variable

### Example

Compute the $df$ of the function $f(x,y) = \ln(1 + x^2 + y^2)$ at the point $(1,1)$. Evaluate this $df$ for $dx = 0.1$, $dy = 0.2$.

Solution: The differential of $f$ at $(x_0, y_0)$ is given by

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

The partial derivatives $f_x$ and $f_y$ are given by

$$f_x(x,y) = \frac{2x}{1 + x^2 + y^2}, \qquad f_y(x,y) = \frac{2y}{1 + x^2 + y^2}.$$

Therefore, $f_x(1,1) = \frac{2}{3} = f_y(1,1)$. Then $df = \frac{2}{3}\,dx + \frac{2}{3}\,dy$.
Evaluating this differential at $dx = 0.1$ and $dy = 0.2$ we obtain

$$df = \frac{2}{3}\frac{1}{10} + \frac{2}{3}\frac{2}{10} = \frac{2}{3}\frac{3}{10} \quad \Rightarrow \quad df = \frac{1}{5}. \qquad \triangleleft$$

## Differential of functions of more than one variable

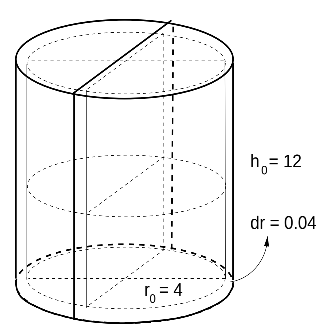
### Example

Use differentials to estimate the amount of aluminum needed to build a closed cylindrical can with internal diameter of $8cm$ and height of $12cm$ if the aluminum is $0.04cm$ thick.

Solution:

The data of the problem is: $h_0 = 12cm$, $r_0 = 4cm$, $dr = 0.04cm$ and $dh = 0.08cm$. The function to consider is the mass of the cylinder, $M = \rho V$, where $\rho = 2.7 gr/cm^3$ is the aluminum density and $V$ is the volume of the cylinder,

$$V(r,h) = \pi r^2 h.$$




The metal to build the can is given by

$$\Delta M = \rho\left[V(r + dr, h + dh) - V(r,h)\right], \quad (\text{recall } dh = 2dr.)$$

## Differential of functions of more than one variable

Example

Use differentials to estimate the amount of aluminum needed to build a closed cylindrical can with internal diameter of $8cm$ and height of $12cm$ if the aluminum is $0.04cm$ thick.

Solution: The metal to build the can is given by

$$\Delta M = \rho \left[ V(r + dr, h + dh) - V(r, h) \right].$$

A linear approximation to $\Delta V = V(r + dr, h + dh) - V(r, h)$ is $dV = V_r\, dr + V_h\, dh$, that is,

$$dV = V_r(r_0, h_0)dr + V_h(r_0, h_0)dh.$$

Since $V(r, h) = \pi r^2 h$, we obtain $dV = 2\pi r_0 h_0\, dr + \pi r_0^2\, dh$.

Therefore, $dV = 16.1\, cm^3$. Since $dM = \rho\, dV$, a linear estimate for the aluminum needed to build the can is $dM = 43.47\, gr$. $\triangleleft$