Directional derivatives and gradient vectors (Sect. 14.5)

- Directional derivative of functions of two variables.
- Partial derivatives and directional derivatives.
- Directional derivative of functions of three variables.
- The gradient vector and directional derivatives.
- Properties of the the gradient vector.

Directional derivative of functions of two variables.

Remark: The directional derivative generalizes the partial derivatives to any direction.

Definition
The *directional derivative* of the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y \rangle$ is given by the limit

$$\big(D_{\mathbf{u}} f\big)_{P_0} = \lim_{t \to 0} \frac{1}{t} \big[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0) \big].$$

Remarks: The line by $\mathbf{r}_0 = \langle x_0, y_0 \rangle$ tangent to $\mathbf{u}$ is $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$.

(a) Equivalently, $\big(D_{\mathbf{u}} f\big)_{P_0} = \lim_{t \to 0} \frac{1}{t} \big[f(\mathbf{r}(t)) - f(\mathbf{r}(0)) \big]$.

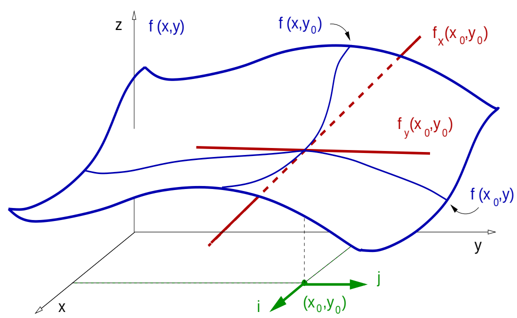
(b) If $\hat{f}(t) = f(\mathbf{r}(t))$, then holds $\big(D_{\mathbf{u}} f\big)_{P_0} = \hat{f}'(0)$.

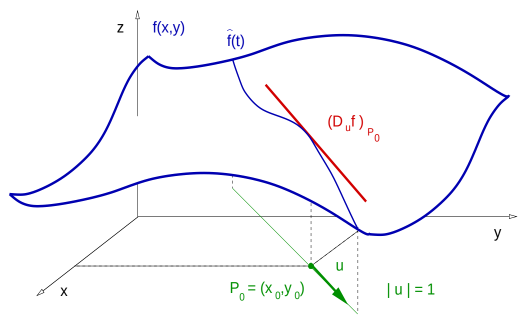
Directional derivatives generalize partial derivatives

Example

The partial derivatives f_x and f_y are particular cases of directional derivatives $\left(D_{\mathbf{u}} f\right)_{P_0} = \lim_{t \to 0} \frac{1}{t}\left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\right]$:

- $\mathbf{u} = \langle 1, 0\rangle = \mathbf{i}$, then $\left(D_{\mathbf{i}} f\right)_{P_0} = f_x(x_0, y_0)$.
- $\mathbf{u} = \langle 0, 1\rangle = \mathbf{j}$, then $\left(D_{\mathbf{j}} f\right)_{P_0} = f_y(x_0, y_0)$.





Directional derivatives generalize partial derivatives

Example

Find the derivative of $f(x, y) = x^2 + y^2$ at $P_0 = (1, 0)$ in the direction of $\theta = \pi/6$ counterclockwise from the x-axis.

Solution: A unit vector in the direction of θ is $\mathbf{u} = \langle \cos(\theta), \sin(\theta)\rangle$. For $\theta = \pi/6$ we get $\mathbf{u} = \langle 1/2, \sqrt{3}/2\rangle$.

The line containing the vector $\mathbf{r}_0 = \langle 1, 0\rangle$ and tangent to $\mathbf{u}$ is

$$\mathbf{r}(t) = \langle 1, 0\rangle + \frac{t}{2}\langle 1, \sqrt{3}\rangle \quad \Rightarrow \quad x(t) = 1 + \frac{t}{2}, \quad y(t) = \frac{\sqrt{3}\,t}{2}.$$

Hence $\hat{f}(t) = f(x(t), y(t))$ is given by

$$\hat{f}(t) = \left(1 + \frac{t}{2}\right)^2 + \frac{3t^2}{4} \quad \Rightarrow \quad \hat{f}(t) = 1 + t + t^2.$$

Since $(D_{\mathbf{u}} f)_{P_0} = \hat{f}'(0)$, and $\hat{f}'(t) = 1 + 2t$, then $(D_{\mathbf{u}} f)_{P_0} = 1$. $\triangleleft$

Directional derivative of functions of two variables

Remark: The condition $|\mathbf{u}| = 1$ in the line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \mathbf{u}\, t$ implies that the parameter t is the distance between the points $(x(t), y(t)) = (x_0 + u_x t, y_0 + u_y t)$ and (x_0, y_0).

In other words: The arc length function of the line is $\ell = t$.

Proof:

$$d = |\langle x - x_0, y - y_0 \rangle| = |\langle u_x t, u_y t \rangle| = |t|\, |\mathbf{u}|, \quad \Rightarrow \quad d = |t|.$$

Equivalently,

$$\mathbf{r}'(t) = \mathbf{u} \quad \Rightarrow \quad \ell = \int_0^t |\mathbf{r}'(\tau)|\, d\tau = \int_0^t d\tau \quad \Rightarrow \quad \ell = t. \qquad \square$$

Remark: The directional derivative $\big(D_{\mathbf{u}} f\big)_{P_0}$ is the pointwise rate of change of f with respect to the distance along the line parallel to $\mathbf{u}$ passing through P_0.

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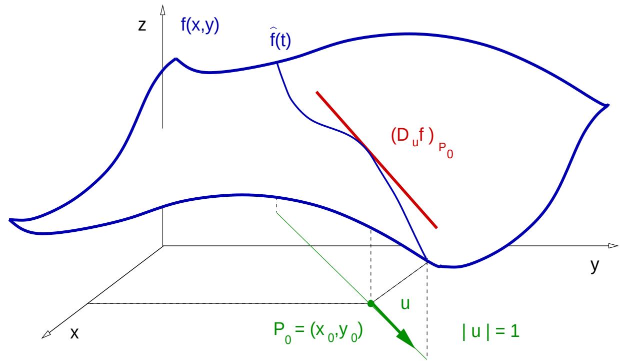
Directional derivative and partial derivatives

Remark: The directional derivative $\left(D_{\mathbf{u}} f\right)_{P_0}$ is the derivative of f along the line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \mathbf{u}\, t$.



Theorem

If the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0)$ and $\mathbf{u} = \langle u_x, u_y \rangle$ is a unit vector, then

$$\left(D_{\mathbf{u}} f\right)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y.$$

Directional derivative and partial derivatives

Proof:

The line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \langle u_x, u_y \rangle\, t$ has parametric equations: $x(t) = x_0 + u_x t$ and $y(t) = y_0 + u_y t$;

Denote f evaluated along the line as $\hat{f}(t) = f(x(t), y(t))$.

Now, on the one hand, $\hat{f}'(0) = \left(D_{\mathbf{u}} f\right)_{P_0}$, since

$$\hat{f}'(0) = \lim_{t \to 0} \frac{1}{t}\left[\hat{f}(t) - \hat{f}(0)\right]$$

$$\hat{f}'(0) = \lim_{t \to 0} \frac{1}{t}\left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\right] = D_{\mathbf{u}} f(x_0, y_0).$$

On the other hand, the chain rule implies:

$$\hat{f}'(0) = f_x(x_0, y_0)\, x'(0) + f_y(x_0, y_0)\, y'(0).$$

Therefore, $\left(D_{\mathbf{u}} f\right)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y$. $\square$

Directional derivative and partial derivatives

Example

Compute the directional derivative of $f(x, y) = \sin(x + 3y)$ at the point $P_0 = (4, 3)$ in the direction of vector $\mathbf{v} = \langle 1, 2 \rangle$.

Solution: We need to find a unit vector in the direction of $\mathbf{v}$.

Such vector is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \dfrac{1}{\sqrt{5}} \langle 1, 2 \rangle$.

We now use the formula $\big(D_{\mathbf{u}} f\big)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y$.

That is, $\big(D_{\mathbf{u}} f\big)_{P_0} = \cos(x_0 + 3y_0)(1/\sqrt{5}) + 3\cos(x_0 + 3y_0)(2/\sqrt{5})$.

Equivalently, $\big(D_{\mathbf{u}} f\big)_{P_0} = (7/\sqrt{5})\cos(x_0 + 3y_0)$.

Then , $\big(D_{\mathbf{u}} f\big)_{P_0} = (7/\sqrt{5})\cos(13)$. $\triangleleft$

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Directional derivative of functions of three variables

Definition

The *directional derivative* of the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0, z_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is given by the limit

$$\big(D_{\mathbf{u}} f\big)_{P_0} = \lim_{t \to 0} \frac{1}{t} \big[f(x_0 + u_x t, y_0 + u_y t, z_0 + u_z t) - f(x_0, y_0, z_0) \big].$$

Theorem

If the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0, z_0)$ and $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is a unit vector, then

$$\big(D_{\mathbf{u}} f\big)_{P_0} = f_x(x_0, y_0, z_0)\, u_x + f_y(x_0, y_0, z_0)\, u_y + f_z(x_0, y_0, z_0)\, u_z.$$

Directional derivative of functions of three variables

Example

Find $\big(D_{\mathbf{u}} f\big)_{P_0}$ for $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at the point $P_0 = (3, 2, 1)$ along the direction given by $\mathbf{v} = \langle 2, 1, 1 \rangle$.

Solution: We first find a unit vector along $\mathbf{v}$,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{6}} \langle 2, 1, 1 \rangle.$$

Then, $\big(D_{\mathbf{u}} f\big)$ is given by $\big(D_{\mathbf{u}} f\big) = (2x) u_x + (4y) u_y + (6z) u_z$.

We conclude, $\big(D_{\mathbf{u}} f\big)_{P_0} = (6)\frac{2}{\sqrt{6}} + (8)\frac{1}{\sqrt{6}} + (6)\frac{1}{\sqrt{6}}$,

that is, $\big(D_{\mathbf{u}} f\big)_{P_0} = \dfrac{26}{\sqrt{6}}$. $\triangleleft$

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The gradient vector and directional derivatives

Remark: The directional derivative of a function can be written in terms of a dot product.

(a) In the case of 2 variable functions: $D_{\mathbf{u}} f = f_x u_x + f_y u_y$

$$D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}, \quad \text{with} \quad \nabla f = \langle f_x, f_y \rangle.$$

(b) In the case of 3 variable functions: $D_{\mathbf{u}} f = f_x u_x + f_y u_y + f_z u_z$,

$$D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}, \quad \text{with} \quad \nabla f = \langle f_x, f_y, f_z \rangle.$$

The gradient vector and directional derivatives

Definition

The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at any point $(x, y) \in D$ is the vector $\nabla f = \langle f_x, f_y \rangle$.

The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at any point $(x, y, z) \in D$ is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$.

Notation:

- For two variable functions: $\nabla f = f_x \, \mathbf{i} + f_y \, \mathbf{j}$.
- For two variable functions: $\nabla f = f_x \, \mathbf{i} + f_y \, \mathbf{j} + f_z \, \mathbf{k}$.

Theorem

If $f : D \subset \mathbb{R}^n \to \mathbb{R}$, *with* $n = 2, 3$, *is a differentiable function and* $\mathbf{u}$ *is a unit vector, then,*

$$D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}.$$

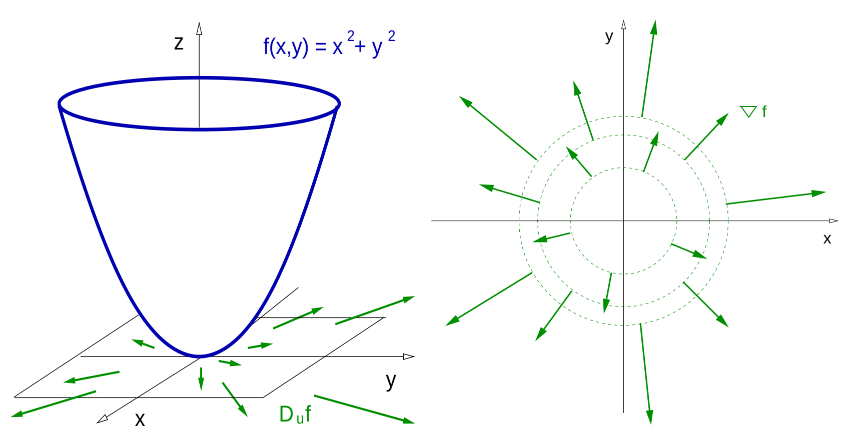
The gradient vector and directional derivatives

Example

Find the gradient vector at any point in the domain of the function $f(x, y) = x^2 + y^2$.

Solution: The gradient is $\nabla f = \langle f_x, f_y \rangle$, that is, $\nabla f = \langle 2x, 2y \rangle$. $\lhd$

Remark:
$\nabla f = 2\mathbf{r}$,
with
$\mathbf{r} = \langle x, y \rangle$.



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Properties of the the gradient vector

Remark: If θ is the angle between ∇f and $\mathbf{u}$, then holds

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} \quad \Rightarrow \quad D_{\mathbf{u}} f = |\nabla f| \cos(\theta).$$

The formula above implies:

- The function f increases the most rapidly when $\mathbf{u}$ is in the direction of ∇f, that is, $\theta = 0$. The maximum increase rate of f is $|\nabla f|$.
- The function f decreases the most rapidly when $\mathbf{u}$ is in the direction of $-\nabla f$, that is, $\theta = \pi$. The maximum decrease rate of f is $-|\nabla f|$.
- Since the function f does not change along level curve or surfaces, that is, $D_{\mathbf{u}} f = 0$, then ∇f is perpendicular to the level curves or level surfaces.

Properties of the the gradient vector

Example

Find the direction of maximum increase of the function $f(x, y) = x^2/4 + y^2/9$ at an arbitrary point (x, y), and also at the points $(1, 0)$ and $(0, 1)$.

Solution: The direction of maximum increase of f is the direction of its gradient vector:

$$\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle.$$

At the points $(1, 0)$ and $(0, 1)$ we obtain, respectively,

$$\nabla f = \left\langle \frac{1}{2}, 0 \right\rangle. \qquad \nabla f = \left\langle 0, \frac{2}{9} \right\rangle.$$

$\triangleleft$

Properties of the the gradient vector.

Example

Given the function $f(x, y) = x^2/4 + y^2/9$, find the equation of a line tangent to a level curve $f(x, y) = 1$ at the point $P_0 = (1, -3\sqrt{3}/2)$.

Solution: We first verify that P_0 belongs to the level curve $f(x, y) = 1$. This is the case, since

$$\frac{1}{4} + \frac{(9)(3)}{4}\,\frac{1}{9} = 1.$$

The equation of the line we look for is

$$\mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t\,\langle v_x, v_y \rangle,$$

where $\mathbf{v} = \langle v_x, v_y \rangle$ is tangent to the level curve $f(x, y) = 1$ at P_0.

Properties of the the gradient vector

Example

Given the function $f(x, y) = x^2/4 + y^2/9$, find the equation of a line tangent to a level curve $f(x, y) = 1$ at the point $P_0 = (1, -3\sqrt{3}/2)$.

Solution: Therefore, $\mathbf{v} \perp \nabla f$ at P_0. Since,

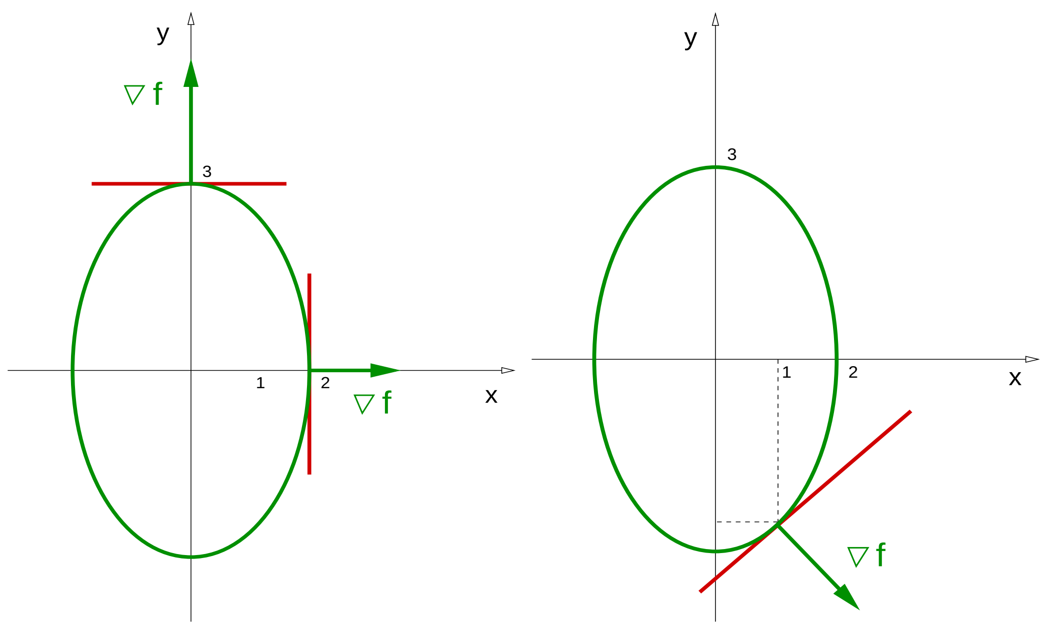
$$\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle \quad \Rightarrow \quad \left(\nabla f\right)_{P_0} = \left\langle \frac{1}{2}, -\frac{2}{9}\frac{3\sqrt{3}}{2} \right\rangle = \left\langle \frac{1}{2}, -\frac{1}{\sqrt{3}} \right\rangle.$$

Therefore,

$$0 = \mathbf{v} \cdot \left(\nabla f\right)_{P_0} \quad \Rightarrow \quad \frac{1}{2} v_x = \frac{1}{\sqrt{3}} v_y \quad \Rightarrow \quad \mathbf{v} = \langle 2, \sqrt{3} \rangle.$$

The line is $\mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t \, \langle 2, \sqrt{3} \rangle$. ◁

Properties of the the gradient vector



$$\nabla f = \left\langle \frac{1}{2}, 0 \right\rangle, \quad \nabla f = \left\langle 0, \frac{2}{9} \right\rangle, \quad \mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t \, \langle 2, \sqrt{3} \rangle.$$

Further properties of the the gradient vector

Theorem

If f, g are differentiable scalar valued vector functions, $g \neq 0$, and $k \in R$ any constant, then holds,

(a) $\nabla(kf) = k\,(\nabla f)$;

(b) $\nabla(f \pm g) = \nabla f \pm \nabla g$;

(c) $\nabla(fg) = (\nabla f)\,g + f\,(\nabla g)$;

(d) $\nabla\Big(\dfrac{f}{g}\Big) = \dfrac{(\nabla f)\,g - f\,(\nabla g)}{g^2}$.