Chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: Chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - Chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - Chain rule for functions defined on a curve in a plane.
  - Chain rule for change of coordinates in a plane.
- Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.
  - Chain rule for functions defined on a curve in space.
  - Chain rule for functions defined on surfaces in space.
  - Chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$

Chain rule for change of coordinates in a line.

Theorem

If the functions $f : [x_0, x_1] \to \mathbb{R}$ and $x : [t_0, t_1] \to [x_0, x_1]$ are differentiable, then the function $\hat{f} : [t_0, t_1] \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(x(t)\big)$ is differentiable and

$$\frac{d\hat{f}}{dt}(t) = \frac{df}{dx}\big(x(t)\big)\,\frac{dx}{dt}(t).$$

Notation:

The equation above is usually written as $\dfrac{d\hat{f}}{dt} = \dfrac{df}{dx}\,\dfrac{dx}{dt}$.

Alternative notations are $\hat{f}'(t) = f'\big(x(t)\big)\,x'(t)$ and $\hat{f}' = f'\,x'$.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$

Chain rule for change of coordinates in a line.

Example

The volume V of a gas balloon depends on the temperature F in Fahrenheit as $V(F) = k\,F^2 + V_0$. Let $F(C) = (9/5)C + 32$ be the temperature in Fahrenheit corresponding to C in Celsius. Find the rate of change $\hat{V}'(C)$.

Solution: Use the chain rule to derivate $\hat{V}(C) = V(F(C))$,

$$\hat{V}'(C) = V'(F)\,F' = 2k\,F\,F' = 2k\left(\frac{9}{5}C + 32\right)\frac{9}{5}.$$

We conclude that $V'(C) = \dfrac{18k}{5}\left(\dfrac{9}{5}C + 32\right)$. $\lhd$

Remark: One could first compute $\hat{V}(C) = k\left(\dfrac{9}{5}\,C + 32\right)^2 + V_0$

and then find the derivative $\hat{V}'(C) = 2k\left(\dfrac{9}{5}\,C + 32\right)\dfrac{9}{5}$.

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Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

The chain rule for functions defined on a curve in a plane.

Theorem

If the functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^2$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ is differentiable and holds

$$\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}\big(\mathbf{r}(t)\big)\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\big(\mathbf{r}(t)\big)\,\frac{dy}{dt}(t).$$

Notation:

The equation above is usually written as $\dfrac{d\hat{f}}{dt} = \dfrac{\partial f}{\partial x}\,\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\,\dfrac{dy}{dt}$.

An alternative notation is $\hat{f}' = f_x\, x' + f_y\, y'$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

Example

Find the rate of change of the function $f(x, y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \sin(t), \cos(2t) \rangle$.

Solution: The rate of change of f along the curve $\mathbf{r}(t)$ is the derivative of $\hat{f}(t) = f(\mathbf{r}(t)) = f(x(t), y(t))$. We do not need to compute $\hat{f}(t) = f(\mathbf{r}(t))$. Instead, the chain rule implies

$$\hat{f}'(t) = f_x(\mathbf{r})\, x' + f_y(\mathbf{r})\, y' = 2x\, x' + 6y^2\, y'.$$

Since $x(t) = \sin(t)$ and $y(t) = \cos(2t)$,

$$\hat{f}'(t) = 2\sin(t)\,\cos(t) + 6\cos^2(2t)\,\big[-2\sin(2t)\big].$$

The result is $\hat{f}'(t) = 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t)$. $\triangleleft$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

The chain rule for change of coordinates in a plane.

Theorem

If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ and the change of coordinate functions $x, y : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t,s)$ and $y(t,s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t,s) = f\big(x(t,s), y(t,s)\big)$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s.$$

Remark: We denote by $f(x,y)$ the function values in the coordinates (x,y), while we denote by $\hat{f}(t,s)$ are the function values in the coordinates (t,s).

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x,y) = x^2 + 3y^2$, in Cartesian coordinates (x,y), find the derivatives of f in polar coordinates (r,θ).

Solution: The relation between Cartesian and polar coordinates is

$$x(r,\theta) = r\cos(\theta), \quad y(r,\theta) = r\sin(\theta).$$

The function f in polar coordinates is $\hat{f}(r,\theta) = f(x(r,\theta), y(r,\theta))$.
The chain rule says $\hat{f}_r = f_x\, x_r + f_y\, y_r$ and $\hat{f}_\theta = f_x\, x_\theta + f_y\, y_\theta$, hence

$$\hat{f}_r = 2x\cos(\theta) + 6y\sin(\theta) \;\Rightarrow\; \hat{f}_r = 2r\cos^2(\theta) + 6r\sin^2(\theta).$$

$$\hat{f}_\theta = -2xr\sin(\theta) + 6yr\cos(\theta),$$

$$\hat{f}_\theta = -2r^2\cos(\theta)\sin(\theta) + 6r^2\cos(\theta)\sin(\theta). \qquad \triangleleft$$

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  - Chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

Chain rule for functions defined on a curve in space.

Theorem

If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ is differentiable and holds

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Notation:

The equation above is usually written as

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Solution: Recall: We do not need to compute

$$\hat{f}(t) = f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$

to find $\hat{f}'$. We only need to use the chain rule formula,

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

$$\hat{f}' = -2x \sin(t) + 3y^2 \cos(t) + 4z^3(3).$$

$$\hat{f}' = -2\cos(t)\sin(t) + 3\sin^2(t)\cos(t) + 4(3)(3^3)t^3. \qquad \triangleleft$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the surface given by functions $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t,s)$ and $y(t,s)$, and $z(t,s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t,s) = f\big(x(t,s), y(t,s), z(t,s)\big)$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s)$ the function values at the surface point with coordinates (t,s).

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Example

Given the function $f(x,y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x,y), find its derivatives on the surface given by $x(t,s) = t+s, \quad y(t,s) = t^2+s^2, \quad z(t,s) = t-s.$

Solution: Recall: We do not need to compute the function

$$\hat{f}(t,s) = f(x(t,s), y(t,s), z(t,s))$$

to obtain the derivatives of f along the surface $x(t,s)$, $y(t,s)$ and $z(t,s)$, which are given by $\hat{f}_t$ and $\hat{f}_s$. We just use the chain rule,

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t \quad \hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

$$\hat{f}_t = 2(t+s) + 6(t^2+s^2)(2t) + 4(t-s),$$

$$\hat{f}_s = 2(t+s) + 6(t^2+s^2)(2s) - 4(t-s). \qquad \triangleleft$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Remark: We describe the surface in the previous example.

Example

Given the surface in parametric form by the equations

$$x(t,s) = t+s, \qquad y(t,s) = t^2+s^2, \qquad z(t,s) = t-s,$$

express that surface as an equation for x, y and z.

Solution: Invert the equations for x and z and obtain t and s,

$$\frac{x+z}{2} = t, \qquad \frac{x-z}{2} = s.$$

We introduce these t and s into the equation for y,

$$y = \Big(\frac{x+z}{2}\Big)^2 + \Big(\frac{x-z}{2}\Big)^2 = \frac{(x^2+z^2+2xz) + (x^2+z^2-2xz)}{4}$$

hence, $y = \dfrac{x^2}{2} + \dfrac{z^2}{2}$, a circular paraboloid along the y axis. $\qquad \triangleleft$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for change of coordinates in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the change of coordinate functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ are differentiable, with $x(t,s,r)$, $y(t,s,r)$, and $z(t,s,r)$, then the function $\hat{f} : \mathbb{R}^3 \to \mathbb{R}$ given by the composition $\hat{f}(t,s,r) = f\big(x(t,s,r), y(t,s,r), z(t,s,r)\big)$ is differentiable and

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s$$
$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r.$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s,r)$ the function values in the coordinates (t,s,r).

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for change of coordinates in space.

Example

Given the function $f(x,y,z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates, find its r-derivative in spherical coordinates (r, θ, ϕ),

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

Solution: Recall: We do not need to compute the function

$$\hat{f}(r,\theta,\phi) = f\big(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)\big).$$

to obtain the r-derivative of f. We just use the chain rule,

$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r = 2x\, x_r + 6y\, y_r + 2z\, z_r$$

$$\hat{f}_r = 2r\cos^2(\phi)\sin^2(\theta) + 6r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta)$$

$$\hat{f}_r = 2r\sin^2(\theta) + 4r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta).$$

We conclude that $\hat{f}_r = 2r + 4r\sin^2(\phi)\sin^2(\theta)$. $\triangleleft$

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- **A formula for implicit differentiation.**

A formula for implicit differentiation

Theorem

If the differentiable function with values $F(x, y)$ defines implicitly the function values $y(x)$ by the equation $F(x, y) = 0$, and if the function $F_y \neq 0$, then y is differentiable and

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Proof: Since $y(x)$ are defined implicitly by the equation

$$F(x, y(x)) = 0,$$

then the function $\hat{F}(x) = F(x, y(x))$ vanishes. Therefore, its derivative vanishes too,

$$0 = \frac{d\hat{F}}{dx} = F_x + F_y\, y' \quad \Rightarrow \quad y' = -\frac{F_x}{F_y}. \qquad \square$$

A formula for implicit differentiation

Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x,y) = 0$, where $F(x,y) = x\,e^y + \cos(x-y)$.

Solution:

The partial derivatives of function F are

$$F_x = e^y - \sin(x-y), \qquad F_y = x\,e^y + \sin(x-y).$$

Therefore, the Theorem above implies

$$y'(x) = \frac{\big[\sin(x-y) - e^y\big]}{\big[x\,e^y + \sin(x-y)\big]}.$$

$\triangleleft$

A formula for implicit differentiation.

Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x,y) = 0$, where $F(x,y) = x\,e^y + \cos(x-y)$.

Solution: We now use the old method.

Since $F(x, y(x)) = x\,e^y + \cos(x-y) = 0$, then differentiating on both sides we get

$$e^y + x\,y'\,e^y - \sin(x-y) - \sin(x-y)(-y') = 0.$$

Reordering terms,

$$y'\,\big[x\,e^y + \sin(x-y)\big] = \sin(x-y) - e^y.$$

We conclude that: $y'(x) = \dfrac{\big[\sin(x-y) - e^y\big]}{\big[x\,e^y + \sin(x-y)\big]}.$ $\triangleleft$