Partial derivatives and continuity

Recall: The following result holds for single variable functions.

**Theorem**

*If the function* \( f : \mathbb{R} \to \mathbb{R} \) *is differentiable, then* \( f \) *is continuous.*

**Proof:**

\[
\lim_{h \to 0} [f(x + h) - f(x)] = \lim_{h \to 0} \left[ \frac{f(x + h) - f(x)}{h} \right] h,
\]

\[
\lim_{h \to 0} [f(x + h) - f(x)] = f'(x) \lim_{h \to 0} h = 0.
\]

That is, \( \lim_{h \to 0} f(x + h) = f(x) \), so \( f \) is continuous.

**Remark:** However, the claim “If \( f_x(x, y) \) and \( f_y(x, y) \) exist, then \( f(x, y) \) is continuous” is false.
Partial derivatives and continuity

**Theorem**

*If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then $f$ is continuous.*

**Remark:**

- This Theorem is not true for the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$.

- There exist functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist but $f$ is not continuous at $(x_0, y_0)$.

**Remark:** This is a bad property for a differentiable function.

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Partial derivatives and continuity

**Remark:** Here is another discontinuous function at $(0, 0)$ having partial derivatives at $(0, 0)$.

**Example**

(a) Show that $f$ is not continuous at $(0, 0);

(b) Find $f_x(0, 0)$ and $f_y(0, 0)$, where

$$f(x, y) = \begin{cases} 
\frac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0), \\
0 & (x, y) = (0, 0).
\end{cases}$$

**Solution:**

(a) Along $x = 0$, $f(0, y) = 0$, so $\lim_{y \to 0} f(0, y) = 0$.

Along the path $x = y$, $f(x, x) = \frac{2x^2}{2x^2} = 1$, so $\lim_{x \to 0} f(x, x) = 1$.

The Two-Path Theorem implies that $\lim_{(x, y) \to (0, 0)} f(x, y) \text{ DNE.}$
Partial derivatives and continuity

Example

(a) Show that \( f \) is not continuous at \((0, 0)\);

\[
f(x, y) = \begin{cases} 
\frac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0), \\
0 & (x, y) = (0, 0).
\end{cases}
\]

(b) Find \( f_x(0, 0) \) and \( f_y(0, 0) \), where

Solution: Recall: \( \lim_{(x,y)\to(0,0)} f(x,y) \) DNE.

(b) The partial derivatives are defined at \((0, 0)\).

\[
f_x(0, 0) = \lim_{h \to 0} \frac{1}{h} [f(0 + h, 0) - f(0, 0)] = \lim_{h \to 0} \frac{1}{h} [0 - 0] = 0.
\]

\[
f_y(0, 0) = \lim_{h \to 0} \frac{1}{h} [f(0, 0 + h) - f(0, 0)] = \lim_{h \to 0} \frac{1}{h} [0 - 0] = 0.
\]

Therefore, \( f_x(0, 0) = f_y(0, 0) = 0 \). △

Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives and continuity.
- **Differentiable functions** \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).
- Differentiability and continuity.
- A primer on differential equations.
Differentiable functions $f: D \subset \mathbb{R}^2 \to \mathbb{R}$

**Recall:** A differentiable function $f: \mathbb{R} \to \mathbb{R}$ at $x_0$ must be approximated by a line $L(x)$ by $(x_0, f(x_0))$ with slope $f'(x_0)$.

The equation of the tangent line is

$$L(x) = f'(x_0) (x - x_0) + f(x_0).$$

The function $f$ is approximated by the line $L$ near $x_0$ means

$$f(x) = L(x) + \epsilon_1 (x - x_0) \quad \text{with} \quad \epsilon_1(x) \to 0 \quad \text{as} \quad x \to x_0.$$

**Remark:** The graph of a differentiable function $f: D \subset \mathbb{R} \to \mathbb{R}$ is approximated by a line at every point in $D$.

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Differentiable functions $f: D \subset \mathbb{R}^2 \to \mathbb{R}$

**Remark:** The idea to define differentiable functions:

The graph of a differentiable function $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in $D$.

We will show next week that the equation of the plane $L$ is

$$L(x, y) = f_x(x_0, y_0) (x - x_0) + f_y(x_0, y_0) (y - y_0) + f(x_0, y_0).$$
Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

**Definition**
Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and an interior point $(x_0, y_0)$ in $D$, let $L$ be the linear function

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

The function $f$ is called **differentiable at** $(x_0, y_0)$ iff the function $f$ is approximated by the linear function $L$ near $(x_0, y_0)$, that is,

$$f(x, y) = L(x, y) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0)$$

where the functions $\epsilon_1$ and $\epsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$. The function $f$ is **differentiable** iff $f$ is differentiable at every interior point of $D$.

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**Remark:** Recalling the linear function $L$ given above,

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0),$$

an equivalent expression for $f$ being differentiable,

$$f(x, y) = L(x, y) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0),$$

is the following: Denote $z = f(x, y)$ and $z_0 = f(x_0, y_0)$, and introduce the increments

$$\Delta z = (z - z_0), \quad \Delta y = (y - y_0), \quad \Delta x = (x - x_0);$$

then, the equation above is

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y.$$  

(Equation used in the textbook to define a differentiable function.)
Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives and continuity.
- Differentiable functions \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).
- **Differentiability and continuity.**
- A primer on differential equations.

**Differentiability and continuity**

**Remark:** We will show in Sect. 14.6 that the graph of a differentiable function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) is approximated by a plane at every point in \( D \).

**Theorem**

If a function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) is differentiable, then \( f \) is continuous.

**Remark:** A simple sufficient condition on a function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) guarantees that \( f \) is differentiable.

**Theorem**

If the partial derivatives \( f_x \) and \( f_y \) of a function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) are continuous in an open region \( R \subset D \), then \( f \) is differentiable in \( R \).
A primer on differential equations

Remark: A differential equation is an equation where the unknown is a function and the function together with its derivatives appear in the equation.

Example
Given a constant $k \in \mathbb{R}$, find all solutions $f : \mathbb{R} \to \mathbb{R}$ to the differential equation

$$f'(x) = k f(x).$$

Solution: Multiply by $e^{-kx}$ the equation above $f'(x) - kf(x) = 0$. The result is $f'(x) e^{-kx} - f(x) ke^{-kx} = 0$.

The left-hand side is a total derivative, $[f(x) e^{-kx}]' = 0$.

The solution of the equation above is $f(x) e^{-kx} = c$, with $c \in \mathbb{R}$.
Therefore, $f(x) = c e^{kx}$. 

\[\triangleright\]
A primer on differential equations

Remark: Often in physical applications appear three differential equations for functions $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, with $n = 2, 3, 4$.

- The Laplace equation: (Gravitation, electrostatics.)
  \[ f_{xx} + f_{yy} + f_{zz} = 0. \]

- The Heat equation: (Heat propagation, diffusion.)
  \[ f_t = k \left( f_{xx} + f_{yy} + f_{zz} \right). \]

- The Wave equation: (Light, sound, gravitation.)
  \[ f_{tt} = v \left( f_{xx} + f_{yy} + f_{zz} \right). \]

Example
Verify that the function \( T(t, x) = e^{-4t} \sin(2x) \) satisfies the one-space dimensional heat equation \( T_t = T_{xx} \).

Solution: We first compute \( T_t \),
\[ T_t = -4e^{-t} \sin(2x). \]

Now compute \( T_{xx} \),
\[ T_x = 2e^{-t} \cos(2x) \quad \Rightarrow \quad T_{xx} = -4e^{-t} \sin(2x) \]

We conclude that \( T_t = T_{xx} \).
A primer on differential equations

Example
Verify that \( f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \) satisfies the Laplace equation: \( f_{xx} + f_{yy} + f_{zz} = 0 \).

Solution: Recall: \( f_x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \). Then,
\[
f_{xx} = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{2} \frac{2x^2}{(x^2 + y^2 + z^2)^{5/2}}.
\]
Denote \( r = \sqrt{x^2 + y^2 + z^2} \), then \( f_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5} \).
Analogously, \( f_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5} \), and \( f_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5} \). Then,
\[
f_{xx} + f_{yy} + f_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0.
\]
We conclude that \( f_{xx} + f_{yy} + f_{zz} = 0 \). \( \bowtie \)

A primer on differential equations

Example
Verify that the function \( f(t, x) = (vt - x)^3 \), with \( v \in \mathbb{R} \), satisfies the one-space dimensional wave equation \( f_{tt} = v^2 f_{xx} \).

Solution: We first compute \( f_{tt} \),
\[
f_t = 3v(vt - x)^2 \quad \Rightarrow \quad f_{tt} = 6v^2(vt - x).
\]
Now compute \( f_{xx} \),
\[
f_x = -3(vt - x)^2 \quad \Rightarrow \quad f_{xx} = 6(vt - x).
\]
Since \( v^2 f_{xx} = 6v^2(vt - x) \), then \( f_{tt} = v^2 f_{xx} \). \( \bowtie \)
A primer on differential equations

Example
Given any \( v \in \mathbb{R} \) and any twice continuously differentiable function \( u : \mathbb{R} \to \mathbb{R} \), verify that \( f(t, x) = u(vt - x) \), satisfies the one-space dimensional wave equation \( f_{tt} = v^2 f_{xx} \).

Solution: We first compute \( f_{tt} \),
\[
f_t = v \ u'(vt - x) \quad \Rightarrow \quad f_{tt} = v^2 \ u''(vt - x).
\]
Now compute \( f_{xx} \),
\[
f_x = -u'(vt - x)^2 \quad \Rightarrow \quad f_{xx} = u''(vt - x).
\]
Since \( v^2 f_{xx} = v^2 u''(vt - x) \), then \( f_{tt} = v^2 f_{xx} \). 
\[
\triangleright
\]