Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Geometrical meaning of partial derivatives.
- The derivative of a function is a new function.
- Higher-order partial derivatives.
- The Mixed Derivative Theorem.
- Examples of implicit partial differentiation.
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Next class:

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Definition

The *partial derivative with respect to x* at a point $(x, y) \in D$ of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ with values $f(x, y)$ is given by

$$f_x(x, y) = \lim_{h \to 0} \frac{1}{h}\big[f(x + h, y) - f(x, y)\big].$$

The *partial derivative with respect to y* at a point $(x, y) \in D$ of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ with values $f(x, y)$ is given by

$$f_y(x, y) = \lim_{h \to 0} \frac{1}{h}\big[f(x, y + h) - f(x, y)\big].$$

Remark:

- To compute $f_x(x, y)$ derivate $f(x, y)$ keeping y constant.
- To compute $f_y(x, y)$ derivate $f(x, y)$ keeping x constant.

Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Remark: To compute $f_x(x, y)$ at (x_0, y_0):

(a) Evaluate the function f at $y = y_0$. The result is a single variable function $\hat{f}(x) = f(x, y_0)$.

(b) Compute the derivative of $\hat{f}$ and evaluate it at $x = x_0$.

(c) The result is $f_x(x_0, y_0)$.

Example

Find $f_x(1, 3)$ for $f(x, y) = x^2 + y^2/4$.

Solution:

(a) $f(x, 3) = x^2 + 9/4$;

(b) $f_x(x, 3) = 2x$;

(c) $f_x(1, 3) = 2$. ◁

Remark: To compute $f_x(x, y)$ derivate $f(x, y)$ keeping y constant.

Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Remark: To compute $f_y(x, y)$ at (x_0, y_0):

(a) Evaluate the function f at $x = x_0$. The result is a single variable function $\tilde{f}(y) = f(x_0, y)$.

(b) Compute the derivative of $\tilde{f}$ and evaluate it at $y = y_0$.

(c) The result is $f_y(x_0, y_0)$.

Example

Find $f_y(1, 3)$ for $f(x, y) = x^2 + y^2/4$.

Solution:

(a) $f(1, y) = 1 + y^2/4$;

(b) $f_y(1, y) = y/2$;

(c) $f_y(1, 3) = 3/2$. ◁

Remark: To compute $f_y(x, y)$ derivate $f(x, y)$ keeping x constant.

Partial derivatives and differentiability (Sect. 14.3)

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- **Geometrical meaning of partial derivatives.**
- The derivative of a function is a new function.
- Higher-order partial derivatives.
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- Examples of implicit partial differentiation.
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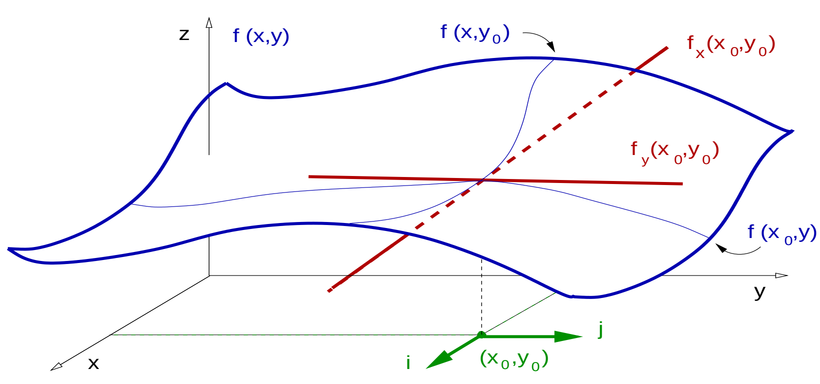
Geometrical meaning of partial derivatives

Remark: $f_x(x_0, y_0)$ is the slope of the line tangent to the graph of $f(x, y)$ containing the point $\big(x_0, y_0, f(x_0, y_0)\big)$ and belonging to a plane parallel to the zx-plane.



Remark: $f_y(x_0, y_0)$ is the slope of the line tangent to the graph of $f(x, y)$ containing the point $\big(x_0, y_0, f(x_0, y_0)\big)$ and belonging to a plane parallel to the zy-plane.

Partial derivatives and differentiability (Sect. 14.3)

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The derivative of a function is a new function

Example

Find the partial derivatives of $f(x, y) = \dfrac{2x - y}{x + 2y}$.

Solution:

$$f_x(x, y) = \frac{2(x + 2y) - (2x - y)}{(x + 2y)^2} \quad \Rightarrow \quad f_x(x, y) = \frac{5y}{(x + 2y)^2}.$$

$$f_y(x, y) = \frac{(-1)(x + 2y) - (2x - y)(2)}{(x + 2y)^2} \Rightarrow f_y(x, y) = -\frac{5x}{(x + 2y)^2}.$$

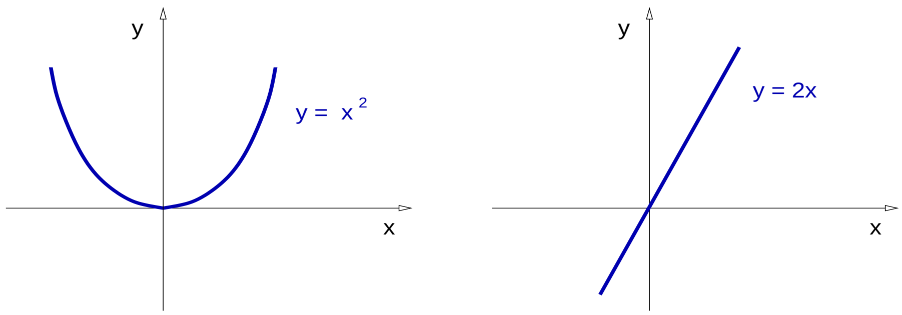
$\lhd$

The derivative of a function is a new function

Recall: The derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is itself a function.

Example

The derivative of function $f(x) = x^2$ at an arbitrary point x is the function $f'(x) = 2x$.



Remark: The same statement is true for partial derivatives.

The partial derivatives of a function are new functions

Definition

Given a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$, the *functions partial derivatives of f* are denoted by f_x and f_y, and they are given by

$$f_x(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f(x+h,y) - f(x,y)\right],$$

$$f_y(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f(x,y+h) - f(x,y)\right].$$

Notation: Partial derivatives of f are denoted in several ways:

$$f_x(x,y), \qquad \frac{\partial f}{\partial x}(x,y), \qquad \partial_x f(x,y).$$

$$f_y(x,y), \qquad \frac{\partial f}{\partial y}(x,y), \qquad \partial_y f(x,y).$$

The partial derivatives of a function are new functions

Remark: The partial derivatives of a paraboloid are planes.

Example

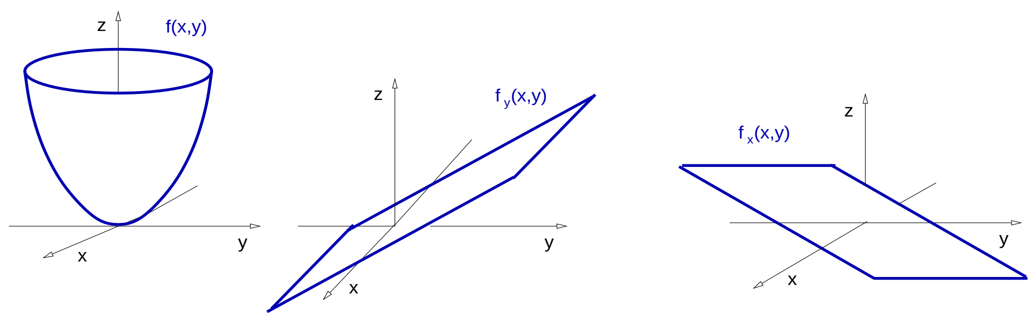
Find the functions partial derivatives of $f(x, y) = x^2 + y^2$.

Solution:

$f_x(x, y) = 2x + 0 \quad \Rightarrow \quad f_x(x, y) = 2x.$

$f_y(x, y) = 0 + 2y \quad \Rightarrow \quad f_y(x, y) = 2y.$

Remark: The partial derivatives of a paraboloid are planes. ◁



The partial derivatives of a function are new functions

Example

Find the partial derivatives of $f(x, y) = x^2 \ln(y)$.

Solution:

$$f_x(x, y) = 2x \ln(y), \qquad f_y(x, y) = \frac{x^2}{y}.$$

◁

Example

Find the partial derivatives of $f(x, y) = x^2 + \frac{y^2}{4}$.

Solution:

$$f_x(x, y) = 2x, \qquad f_y(x, y) = \frac{y}{2}.$$

◁

Partial derivatives and differentiability (Sect. 14.3)

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Higher-order partial derivatives

Remark: Higher derivatives of a function are partial derivatives of its partial derivatives. The second partial derivatives of $f(x, y)$ are:

$$f_{xx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$

$$f_{yy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right],$$

$$f_{xy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x,y+h) - f_x(x,y)\right],$$

$$f_{yx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x+h,y) - f_y(x,y)\right].$$

Notation: f_{xx}, $\dfrac{\partial^2 f}{\partial x^2}$, $\partial_{xx} f$, and f_{xy}, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\partial_{xy} f$.

Higher-order partial derivatives.

Example

Find all second order derivatives of the function
$f(x, y) = x^3 e^{2y} + 3y$.

Solution:

$$f_x(x, y) = 3x^2 e^{2y}, \qquad f_y(x, y) = 2x^3 e^{2y} + 3.$$

$$f_{xx}(x, y) = 6x e^{2y}, \qquad f_{yy}(x, y) = 4x^3 e^{2y}.$$

$$f_{xy} = 6x^2 e^{2y}, \qquad f_{yx} = 6x^2 e^{2y}.$$

$\triangleleft$

Partial derivatives and differentiability (Sect. 14.3).

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The Mixed Derivative Theorem

Remark: Higher-order partial derivatives sometimes commute.

Theorem

If the partial derivatives f_x, f_y, f_{xy} and f_{yx} of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ exist and all are continuous functions, then holds

$$f_{xy} = f_{yx}.$$

Example

Find f_{xy} and f_{yx} for $f(x, y) = \cos(xy)$.

Solution:

$$f_x = -y \sin(xy), \qquad f_{xy} = -\sin(xy) - yx \cos(xy).$$

$$f_y = -x \sin(xy), \qquad f_{yx} = -\sin(xy) - xy \cos(xy). \qquad \triangleleft$$

Partial derivatives and differentiability (Sect. 14.3)

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Examples of implicit partial differentiation

Remark: Implicit differentiation rules for partial derivatives are similar to those for functions of one variable.

Example

Find $\partial_x z(x, y)$ of the function z defined implicitly by the equation $xyz + e^{2z/y} + \cos(z) = 0$.

Solution: Compute the x-derivative on both sides of the equation,

$$yz + xy\,(\partial_x z) + \frac{2}{y}\,(\partial_x z) e^{2z/y} - (\partial_x z)\,\sin(z) = 0.$$

Compute $\partial_x z$ as a function of x, y and $z(x, y)$, as follows,

$$(\partial_x z)\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big] = -yz.$$

We obtain: $(\partial_x z) = -\dfrac{yz}{\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big]}$. ◁

Examples of implicit partial differentiation

Remark: Implicit differentiation rules for partial derivatives are similar to those for functions of one variable.

Example

Find $\partial_y z(x, y)$ of the function z defined implicitly by the equation $xyz + e^{2z/y} + \cos(z) = 0$.

Solution: Compute the y-derivative on both sides of the equation,

$$xz + xy\,(\partial_y z) + \Big(\frac{2}{y}\,(\partial_y z) - \frac{2}{y^2}\,z\Big) e^{2z/y} - (\partial_y z)\,\sin(z) = 0.$$

Compute $\partial_y z$ as a function of x, y and $z(x, y)$, as follows,

$$(\partial_y z)\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big] = -xz + \frac{2}{y^2}\,z\,e^{2z/y},$$

We obtain: $(\partial_y z) = \dfrac{\big[-xz + \frac{2}{y^2}\,z\,e^{2z/y}\big]}{\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big]}$. ◁

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Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$

Definition

The *partial derivative with respect to* x_i at a point $(x_1, \cdots, x_n) \in D$ of a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$ and $i = 1, \cdots, n$, is given by

$$f_{x_i} = \lim_{h \to 0} \frac{1}{h} \big[f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n) \big].$$

Remark: To compute f_{x_i} derivate f with respect to x_i keeping all other variables x_j constant.

Notation: $f_{x_i}, \qquad f_i, \qquad \dfrac{\partial f}{\partial x_i}, \qquad \partial_{x_i} f, \qquad \partial_i f.$

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$

Example

Compute all first partial derivatives of the function
$\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$.

Solution:

$$\phi_x = -\frac{1}{2} \frac{2x}{(x^2 + y^2 + z^2)^{3/2}} \quad \Rightarrow \quad \phi_x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}.$$

Analogously, the other partial derivatives are given by

$$\phi_y = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \phi_z = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}.$$

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Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$

Example

Verify that $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation: $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$.

Solution: Recall: $\phi_x = -x/(x^2 + y^2 + z^2)^{3/2}$. Then,

$$\phi_{xx} = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{2} \frac{2x^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $\phi_{xx} = -\dfrac{1}{r^3} + \dfrac{3x^2}{r^5}$.

Analogously, $\phi_{yy} = -\dfrac{1}{r^3} + \dfrac{3y^2}{r^5}$, and $\phi_{zz} = -\dfrac{1}{r^3} + \dfrac{3z^2}{r^5}$. Then,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5}.$$

We conclude that $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$. $\lhd$