Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2)

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

The limit of functions of several variables.

Definition

The *limit* of the function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, at the point $\hat{P} \in \mathbb{R}^n$ is the number $L \in \mathbb{R}$, denoted as $\lim_{P \to \hat{P}} f(P) = L$, iff for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |P - \hat{P}| < \delta \quad \Rightarrow \quad |f(P) - L| < \epsilon.$$

Remarks:

(a) In Cartesian coordinates $P = (x_1, \cdots, x_n)$, $\hat{P} = (\hat{x}_1, \cdots, \hat{x}_n)$. Then, $|P - \hat{P}|$ is the distance between points in $\mathbb{R}^n$,

$$|P - \hat{P}| = |\overrightarrow{P\hat{P}}| = \sqrt{(x_1 - \hat{x}_1)^2 + \cdots + (x_n - \hat{x}_n)^2}.$$

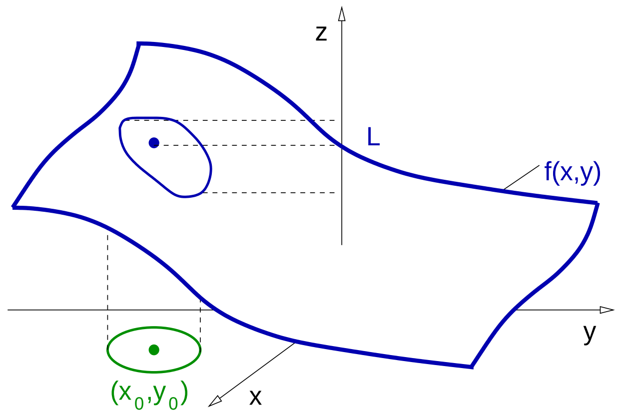
(b) $|f(P) - L|$ is the absolute value of real numbers.

The limit of functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Idea of the limit definition: Consider $f : \mathbb{R}^2 \to \mathbb{R}$. Then,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

means that the closer (x, y) is to (x_0, y_0) then the closer the value of $f(x, y)$ is to L.



Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2)

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- **Example: Computing a limit by the definition.**
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Computing a limit by the definition

Example

Use the definition of limit to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2yx^2}{x^2+y^2}$.

Solution: The function $f(x,y) = \dfrac{2yx^2}{x^2+y^2}$ is not defined at $(0,0)$.

First: Guess what the limit L is.
Along the line $x = 0$ the function is $f(0,y) = \dfrac{0}{y^2} = 0$.

Therefore, if L exists, it must be $L = 0$. Given ϵ, find δ.

Fix any number $\epsilon > 0$. Given that ϵ, find a number $\delta > 0$ such that

$$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta \quad \Rightarrow \quad \left|\frac{2yx^2}{x^2+y^2} - 0\right| < \epsilon.$$

Computing a limit by the definition

Example

Use the definition of limit to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2yx^2}{x^2+y^2}$.

Solution: Given any $\epsilon > 0$, find a number $\delta > 0$ such that

$$\sqrt{x^2+y^2} < \delta \quad \Rightarrow \quad \left|\frac{2yx^2}{x^2+y^2}\right| < \epsilon.$$

Recall: $x^2 \leqslant x^2 + y^2$, that is, $\dfrac{x^2}{x^2+y^2} \leqslant 1$. Then

$$\left|\frac{2yx^2}{x^2+y^2}\right| = (2|y|)\,\frac{x^2}{x^2+y^2} \leqslant 2|y| = 2\sqrt{y^2} \leqslant 2\sqrt{x^2+y^2}.$$

Choose $\delta = \epsilon/2$. If $\sqrt{x^2+y^2} < \delta$, then $\left|\dfrac{2yx^2}{x^2+y^2}\right| < 2\delta = \epsilon$.
We conclude that $L = 0$. $\triangleleft$

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2)

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- **Properties of limits of functions.**
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Properties of limits of functions

Theorem

If $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, satisfying the conditions $\lim_{P \to \hat{P}} f(P) = L$ and $\lim_{P \to \hat{P}} g(P) = M$, then holds

(a) $\lim_{P \to \hat{P}} f(P) \pm g(P) = L \pm M$;

(b) *If* $k \in \mathbb{R}$, *then* $\lim_{P \to \hat{P}} kf(P) = kL$;

(c) $\lim_{P \to \hat{P}} f(P)\, g(P) = LM$;

(d) *If* $M \neq 0$, *then* $\lim_{P \to \hat{P}} \Big[\dfrac{f(P)}{g(P)}\Big] = \dfrac{L}{M}$.

(e) *If* $k \in \mathbb{Z}$ *and* $s \in \mathbb{N}$, *then* $\lim_{P \to \hat{P}} \big[f(P)\big]^{r/s} = L^{r/s}$.

Remark: The Theorem above implies: If $f = R/S$ is the quotient of two polynomials with $S(\hat{P}) \neq 0$, then $\lim_{P \to \hat{P}} f(P) = f(\hat{P})$.

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2)

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- **Examples: Computing limits of simple functions.**
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Limits of R/S at $\hat{P}$ where $S(\hat{P}) \neq 0$ are simple to find

Example

Compute $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^2 + 2y - x}{\sqrt{x-y}}$.

Solution: The function above is a rational function in x and y and its denominator is defined and does not vanish at $(2,1)$. Therefore

$$\lim_{(x,y)\to(2,1)} \frac{x^2 + 2y - x}{\sqrt{x-y}} = \frac{2^2 + 2(1) - 2}{\sqrt{2-1}},$$

that is,

$$\lim_{(x,y)\to(2,1)} \frac{x^2 + 2y - x}{\sqrt{x-y}} = 4.$$

$\lhd$

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- **Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.**
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$

Definition

A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, is called *continuous at* $\hat{P} \in D$ iff holds $\lim_{P \to \hat{P}} f(P) = f(\hat{P})$.

Remarks:

- The definition above says three things:
  (a) $f(\hat{P})$ is defined;
  (b) $\lim_{P \to \hat{P}} f(P) = L$ exists;
  (c) $f(\hat{P}) = L$.
- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is *continuous* iff f is continuous at every point in D.
- Continuous functions have graphs without holes or jumps.

Continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$

Example

- Polynomial functions are continuous in $\mathbb{R}^n$.

  For example, $P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2$.

- Rational functions $f = R/S$ are continuous on their domain.

  For example, $f(x, y) = \dfrac{x^2 + 3y - x^2 y^2 + y^4}{x^2 - y^2}$, with $x \neq \pm y$.

- Composition of continuous functions are continuous.

  For example, $f(x, y) = e^{\cos(x^2+y^2)}$.

Continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$

Example

Compute $\displaystyle\lim_{(x,y)\to(\sqrt{\pi},0)} e^{\cos(x^2+y^2)}$.

Solution:

The function $f(x, y) = e^{\cos(x^2+y^2)}$ is continuous for all $(x, y) \in \mathbb{R}^2$. Therefore,

$$\lim_{(x,y)\to(\sqrt{\pi},0)} e^{\cos(x^2+y^2)} = e^{\cos(\pi+0)} = e^{-1},$$

that is,

$$\lim_{(x,y)\to(\sqrt{\pi},0)} e^{\cos(x^2+y^2)} = \frac{1}{e}.$$

$\triangleleft$

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2)

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- **Computing limits of non-continuous functions:**
  - **Two-path test for the non-existence of limits.**
  - The sandwich test for the **existence** of limits.

Two-path test for the non-existence of limits

Theorem

If a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, has two different limits along two different paths as P approaches $\hat{P}$, then $\lim_{P \to \hat{P}} f(P)$ does not exist.

Remark: Consider the case $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

If

(a) $f(x, y) \to L_1$ along a path C_1 as $(x, y) \to (x_0, y_0)$,

(b) $f(x, y) \to L_2$ along a path C_2 as $(x, y) \to (x_0, y_0)$,

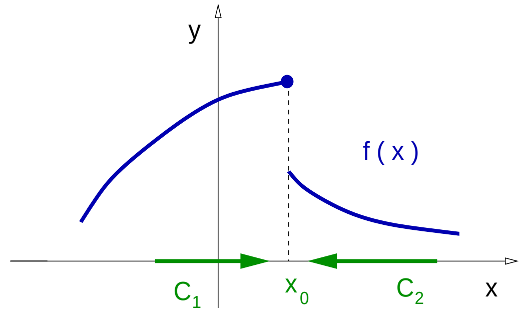
(c) $L_1 \neq L_2$,

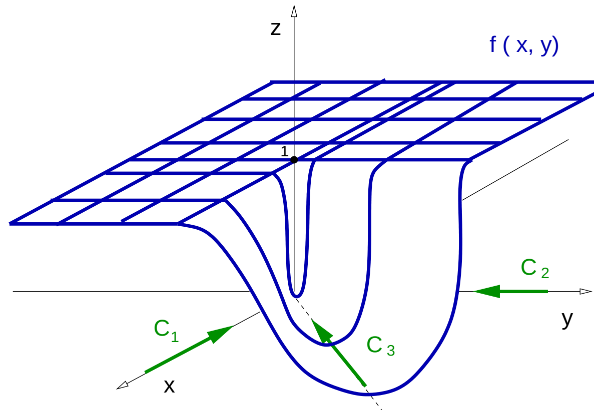
then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Remark: When side limits do not agree, the limit does not exist.

Two-path test for the non-existence of limits

Remark: When side limits do not agree, the limit does not exist.





Two-path test for the non-existence of limits

Example

Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+2y^2}$.

Solution: $f(x,y) = \dfrac{3x^2}{(x^2+2y^2)}$ is not continuous at $(0,0)$.

We try to show that the limit above does not exist.

If path C_1 is the x-axis, $(y=0)$, then,

$$f(x,0) = \frac{3x^2}{x^2} = 3, \quad \Rightarrow \quad \lim_{(x,0)\to(0,0)} f(x,0) = 3.$$

If path C_2 is the y-axis, $(x=0)$, then,

$$f(0,y) = 0, \quad \Rightarrow \quad \lim_{(0,y)\to(0,0)} f(0,y) = 0.$$

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+2y^2}$ does not exist. $\triangleleft$

Two-path test for the non-existence of limits

Remark:
In the example above one could compute the limit for arbitrary lines, that is, C_m given by $y = mx$, with m a constant.
That is,

$$f(x, mx) = \frac{3x^2}{x^2 + 2m^2x^2} = \frac{3}{1 + 2m^2}.$$

The limits along these paths are:

$$\lim_{(x,mx)\to(0,0)} f(x, mx) = \frac{3}{1 + 2m^2}$$

which are different for each value of m.

This agrees with our conclusion: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2 + 2y^2}$ DNE.

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2)

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- **Computing limits of non-continuous functions:**
  - Two-path test for the **non-existence** of limits.
  - **The sandwich test for the existence of limits.**

The sandwich test for the existence of limits

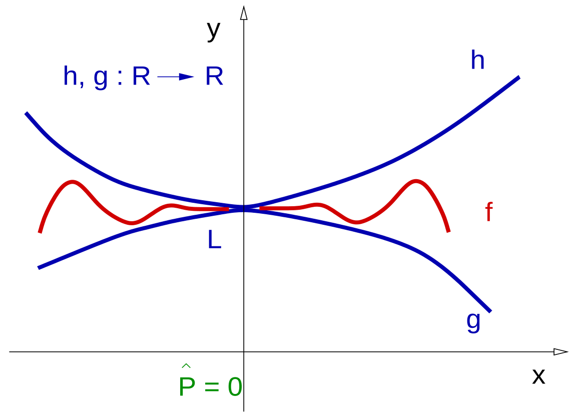
Theorem

If functions $f, g, h : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, satisfy:

(a) $g(P) \leqslant f(P) \leqslant h(P)$ *for all* P *near* $\hat{P} \in D$;

(b) $\lim_{P \to \hat{P}} g(P) = L = \lim_{P \to \hat{P}} h(P)$;

then $\lim_{P \to \hat{P}} f(P) = L$.



The sandwich test for the existence of limits

Example

Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$.

Solution: $f(x,y) = \dfrac{x^2 y}{x^2 + y^2}$ is not continuous at $(0,0)$.

The Two-Path Theorem does not prove non-existence of the limit.

Because: Consider paths C_m given by $y = mx$, with $m \in \mathbb{R}$. Then

$$f(x, mx) = \frac{x^2 mx}{x^2 + m^2 x^2} = \frac{mx}{1 + m^2},$$

which implies $\displaystyle\lim_{(x,mx)\to(0,0)} f(x, mx) = 0, \quad \forall\, m \in \mathbb{R}.$

We cannot conclude that the limit does not exist.
We cannot conclude that the limit exists.

The sandwich test for the existence of limits

Example

Compute $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2}$.

Solution: Recall: $\dfrac{x^2}{x^2+y^2} \leqslant 1$, for all $(x,y) \neq (0,0)$.

So, $\left|\dfrac{x^2 y}{x^2+y^2}\right| = |y| \left(\dfrac{x^2}{x^2+y^2}\right) \leqslant |y|$, for all $(x,y) \neq (0,0)$. So,

$$\left|\frac{x^2 y}{x^2+y^2}\right| \leqslant |y| \quad \Leftrightarrow \quad -|y| \leqslant \frac{x^2 y}{x^2+y^2} \leqslant |y|.$$

Since $\lim_{y\to 0} \pm|y| = 0$, the Sandwich Theorem with functions $g = -|y|$, $h = |y|$, implies

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0. \qquad \triangleleft$$