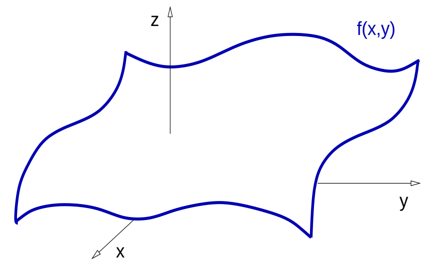
Scalar functions of several variables (Sect. 14.1)

- Functions of several variables.
- On open, closed sets.
- Functions of two variables:
  - Graph of the function.
  - Level curves, contour curves.
- Functions of three variables.
  - Graphs, level surfaces.



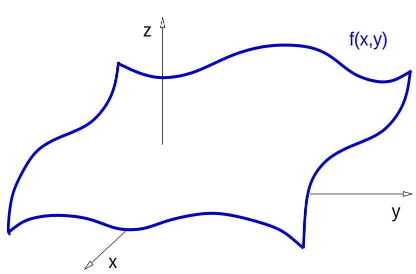
Scalar functions of several variables

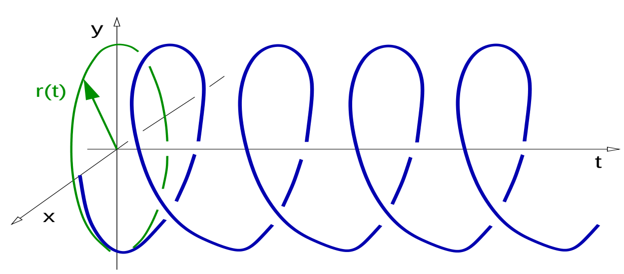
Definition

A *scalar function of n variables* is a function $f : D \subset \mathbb{R}^n \to R \subset \mathbb{R}$, where $n \in \mathbb{N}$, the set D is called the *domain* of the function, and the set R is called the *range* of the function.

Remark:

Comparison between $f : \mathbb{R}^2 \to \mathbb{R}$ with $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$.





Functions of several variables

Example

- An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature T of a plane surface, say a table. Each point (x, y) on the table is associated with a number, its temperature $T(x, y)$.
- An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature T of this room. Each point (x, y, z) in the room is associated with a number, its temperature $T(x, y, z)$.
- Another example of a scalar function of three variables is the speed of the air in the room.
- An example of a vector-valued function of three variables, $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$, is the velocity of the air in the room.

$\triangleleft$

Scalar functions of several variables

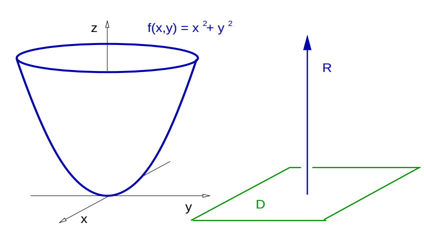
Example

Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is defined.

Solution:

The function $f(x, y) = x^2 + y^2$ is defined for all points $(x, y) \in \mathbb{R}^2$, therefore, $D = \mathbb{R}^2$.

Since $f(x, y) = x^2 + y^2 \geqslant 0$ for all $(x, y) \in D$, then the range space is $R = [0, \infty)$. $\triangleleft$



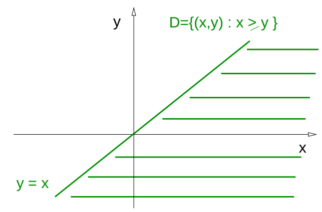
Scalar functions of several variables

Example

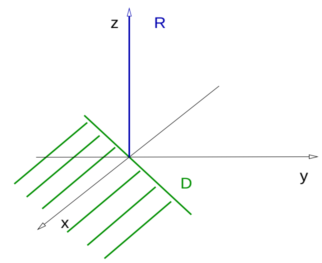
Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x,y) = \sqrt{x-y}$ is defined.

Solution:

The function f is defined for points $(x,y) \in \mathbb{R}^2$ such that $x - y \geqslant 0$. So, $D = \{(x,y) \in \mathbb{R}^2 \ : \ x \geqslant y\}$.



Since $f(x,y) = \sqrt{x-y} \geqslant 0$ for all $(x,y) \in D$, the range space is $R = [0, \infty)$. $\triangleleft$



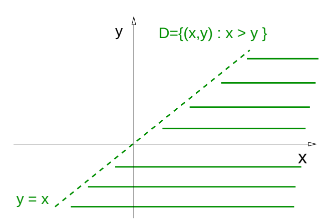
Scalar functions of several variables

Example

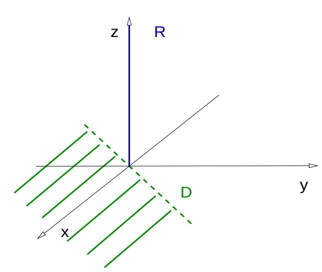
Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x,y) = 1/\sqrt{x-y}$ is defined.

Solution:

The function f is defined for points $(x,y) \in \mathbb{R}^2$ such that $x - y > 0$. So, $D = \{(x,y) \in \mathbb{R}^2 \ : \ x > y\}$.



Since $f(x,y) = 1/\sqrt{x-y} \geqslant 0$ for all $(x,y) \in D$, the range space is $R = (0, \infty)$. $\triangleleft$



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On open and closed sets in $\mathbb{R}^n$

Remark: We first generalize from $\mathbb{R}^3$ to $\mathbb{R}^n$ the definition of a ball of radius r centered at $\hat{P}_c$.

Definition
An *open ball* of radius $r > 0$ centered at $\hat{P}_c = (\hat{x}_1, \cdots, \hat{x}_n)$ is the set in $\mathbb{R}^n$, with $n \in \mathbb{N}$, given by

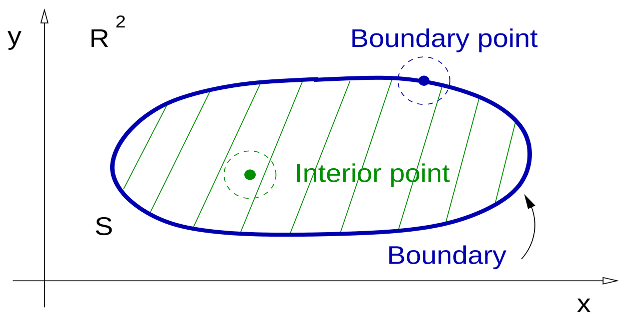
$$B_r(\hat{P}_c) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \ : \ (x_1 - \hat{x}_1)^2 + \cdots + (x_n - \hat{x}_n)^2 < r^2\}.$$

Remark: An open ball $B_r(\hat{P}_c)$ contains the points *inside* a sphere of radius r centered at $\hat{P}_c$ *without* the points of the sphere.

On open and closed sets in $\mathbb{R}^n$

Definition

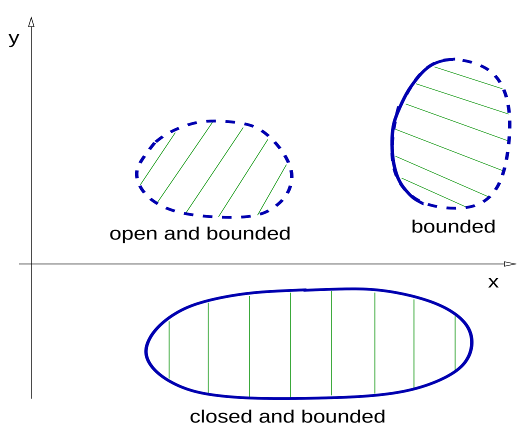
A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called an *interior point* iff there is a ball $B_r(P) \subset S$. A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called a *boundary point* iff every ball $B_r(P)$ contains points in S and points outside S. The *boundary* of a set S is the set of all boundary points of S.

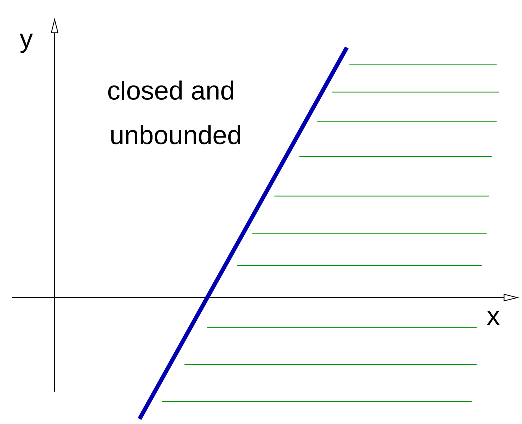


On open and closed sets in $\mathbb{R}^n$

Definition

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in S is an interior point. The set S is called *closed* iff S contains its boundary. A set S is called *bounded* iff S is contained in ball, otherwise S is called *unbounded*.





On open and closed sets in $\mathbb{R}^n$

Example

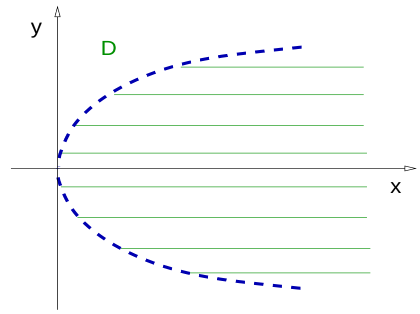
Find and describe the maximum domain of the function $f(x, y) = \ln(x - y^2)$.

Solution:

The maximum domain of f is

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x > y^2\}.$$

D is an open, unbounded set. $\triangleleft$



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The graph of a function of two variables is a surface in $\mathbb{R}^3$

Definition
The *graph* of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the set of all points (x, y, z) in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$. The graph of a function f is also called the surface $z = f(x, y)$.

Example
Draw the graph of $f(x, y) = x^2 + y^2$.

Solution: The graph of f is the surface $z = x^2 + y^2$. This is a paraboloid along the z axis.

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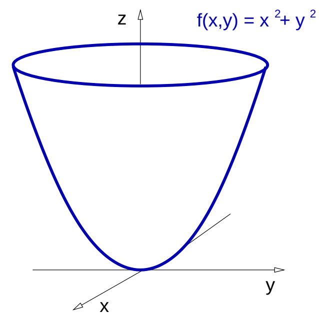

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Level curves, contour curves

Definition

The *contour curves* of a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ are the curves in $\mathbb{R}^3$ given by the equation

$$f(x,y) = k, \qquad z = k, \qquad (x,y) \in D, \qquad k \in R.$$

The *level curves* of the function f are the curves in the domain $D \subset \mathbb{R}^2$ given by the equation

$$f(x,y) = k, \qquad (x,y) \in D, \qquad k \in R.$$

Remark: Contour curves are the intersection of the graph of f with horizontal planes $z = k$.

Remark: Level curves are the vertical translation of contour curves to the function domain.

Level curves, contour curves.

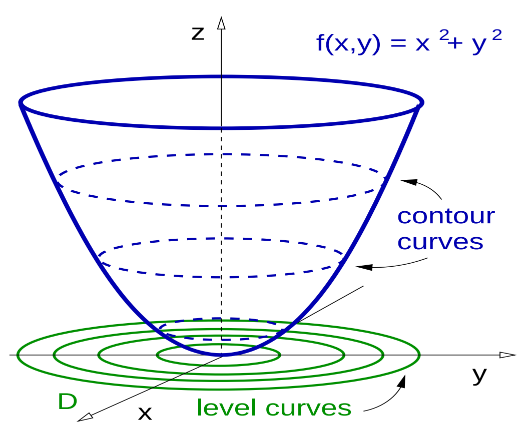
Example

Find and draw few level curves and contour curves for the function $f(x,y) = x^2 + y^2$.

Solution:

The level curves are solutions of the equation $x^2 + y^2 = k$ with $k \geqslant 0$. These curves are circles of radius $\sqrt{k}$ in $D = \mathbb{R}^2$.

The contour curves are the circles $\{(x,y,z) \ : \ x^2+y^2 = k, \ z = k\}$.



Level curves, contour curves

Example

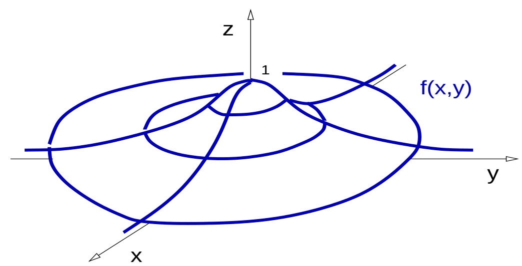
Find the maximum domain, range of, and graph the function $f(x,y) = \dfrac{1}{1+x^2+y^2}$.

Solution:

Since the denominator never vanishes, hence $D = \mathbb{R}^2$.

Since $0 < \dfrac{1}{1+x^2+y^2} \leqslant 1$, the range of f is $R = (0,1]$.

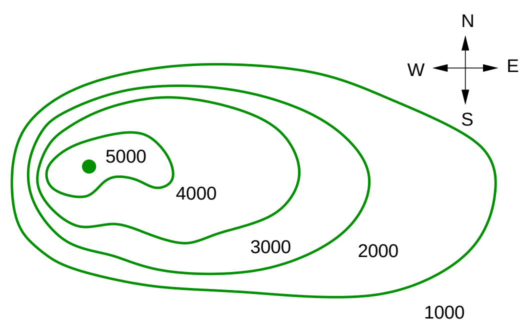
The contour curves are circles on horizontal planes in $(0,1]$. $\triangleleft$

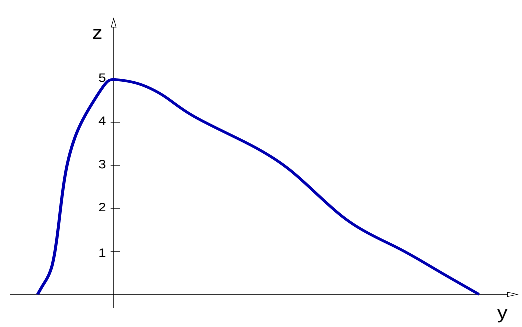


Level curves, contour curves

Example

Given the topographic map in the figure, which way do you choose to the summit?





Solution:

From the east side. $\triangleleft$

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Scalar functions of three variables

Definition

The *graph* of a scalar function of three variables, $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$, is the set of points in $\mathbb{R}^4$ of the form $(x, y, z, f(x, y, z))$ for every $(x, y, z) \in D$.

Remark:

The graph a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ requires four space dimensions. We cannot picture such graph.

Definition

The *level surfaces* of a function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$ are the surfaces in the domain $D \subset \mathbb{R}^3$ of f solutions of the equation $f(x, y, z) = k$, where $k \in R$ is a constant in the range of f.

Scalar functions of three variables

Example

Draw one level surface of the function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$
$f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

Solution:

The domain of f is $D = \mathbb{R}^3$, the range is $R = (0, \infty)$. For $k > 0$ the level surfaces $f(x, y, z) = k$ are

$$x^2 + y^2 + z^2 = \frac{1}{k},$$

spheres radius $R = \dfrac{1}{\sqrt{k}}$. $\triangleleft$