Scalar functions of several variables (Sect. 14.1)

- Functions of several variables.
- On open, closed sets.
- Functions of two variables:
  - Graph of the function.
  - Level curves, contour curves.
- Functions of three variables.
  - Graphs, level surfaces.
Functions of several variables

Example

- An example of a scalar-valued function of two variables, \( T : \mathbb{R}^2 \rightarrow \mathbb{R} \) is the temperature \( T \) of a plane surface, say a table. Each point \((x, y)\) on the table is associated with a number, its temperature \( T(x, y) \).

- An example of a scalar-valued function of three variables, \( T : \mathbb{R}^3 \rightarrow \mathbb{R} \) is the temperature \( T \) of this room. Each point \((x, y, z)\) in the room is associated with a number, its temperature \( T(x, y, z) \).

- Another example of a scalar function of three variables is the speed of the air in the room.

- An example of a vector-valued function of three variables, \( \mathbf{v} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), is the velocity of the air in the room.

Scalar functions of several variables

Example

Find the maximum domain \( D \) and range \( R \) sets where the function \( f : D \subset \mathbb{R}^2 \rightarrow R \subset \mathbb{R} \) given by \( f(x, y) = x^2 + y^2 \) is defined.

Solution:

The function \( f(x, y) = x^2 + y^2 \) is defined for all points \((x, y)\) \( \in \mathbb{R}^2 \), therefore, \( D = \mathbb{R}^2 \).

Since \( f(x, y) = x^2 + y^2 \geq 0 \) for all \((x, y)\) \( \in D \), then the range space is \( R = [0, \infty) \).
Scalar functions of several variables

Example
Find the maximum domain $D$ and range $R$ sets where the function $f: D \subset \mathbb{R}^2 \rightarrow R \subset \mathbb{R}$ given by $f(x, y) = \sqrt{x - y}$ is defined.

Solution:

The function $f$ is defined for points $(x, y) \in \mathbb{R}^2$ such that $x - y \geq 0$. So, $D = \{(x, y) \in \mathbb{R}^2 : x \geq y\}$.

Since $f(x, y) = \sqrt{x - y} \geq 0$ for all $(x, y) \in D$, the range space is $R = [0, \infty)$.

Scalar functions of several variables

Example
Find the maximum domain $D$ and range $R$ sets where the function $f: D \subset \mathbb{R}^2 \rightarrow R \subset \mathbb{R}$ given by $f(x, y) = \frac{1}{\sqrt{x - y}}$ is defined.

Solution:

The function $f$ is defined for points $(x, y) \in \mathbb{R}^2$ such that $x - y > 0$. So, $D = \{(x, y) \in \mathbb{R}^2 : x > y\}$.

Since $f(x, y) = \frac{1}{\sqrt{x - y}} \geq 0$ for all $(x, y) \in D$, the range space is $R = (0, \infty)$. 
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On open and closed sets in $\mathbb{R}^n$

**Remark:** We first generalize from $\mathbb{R}^3$ to $\mathbb{R}^n$ the definition of a ball of radius $r$ centered at $\hat{P}_c$.

**Definition**
An *open ball* of radius $r > 0$ centered at $\hat{P}_c = (\hat{x}_1, \cdots, \hat{x}_n)$ is the set in $\mathbb{R}^n$, with $n \in \mathbb{N}$, given by

$$B_r(\hat{P}_c) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : (x_1 - \hat{x}_1)^2 + \cdots + (x_n - \hat{x}_n)^2 < r^2\}.$$

**Remark:** An open ball $B_r(\hat{P}_c)$ contains the points *inside* a sphere of radius $r$ centered at $\hat{P}_c$ *without* the points of the sphere.
On open and closed sets in $\mathbb{R}^n$

**Definition**

A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called an **interior point** iff there is a ball $B_r(P) \subset S$. A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called a **boundary point** iff every ball $B_r(P)$ contains points in $S$ and points outside $S$. The **boundary** of a set $S$ is the set of all boundary points of $S$.

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On open and closed sets in $\mathbb{R}^n$

**Definition**

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called **open** iff every point in $S$ is an interior point. The set $S$ is called **closed** iff $S$ contains its boundary. A set $S$ is called **bounded** iff $S$ is contained in ball, otherwise $S$ is called **unbounded**.
On open and closed sets in $\mathbb{R}^n$

Example
Find and describe the maximum domain of the function $f(x, y) = \ln(x - y^2)$.

Solution:
The maximum domain of $f$ is

$$D = \{(x, y) \in \mathbb{R}^2 : x > y^2\}.$$  

$D$ is an open, unbounded set. ◄

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The graph of a function of two variables is a surface in $\mathbb{R}^3$

**Definition**

The *graph* of a function $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is the set of all points $(x, y, z)$ in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$. The graph of a function $f$ is also called the surface $z = f(x, y)$.

**Example**

Draw the graph of $f(x, y) = x^2 + y^2$.

**Solution:** The graph of $f$ is the surface $z = x^2 + y^2$. This is a paraboloid along the $z$ axis.

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Level curves, contour curves

Definition
The contour curves of a function \( f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R} \) are the curves in \( \mathbb{R}^3 \) given by the equation

\[
f(x, y) = k, \quad z = k, \quad (x, y) \in D, \quad k \in R.
\]

The level curves of the function \( f \) are the curves in the domain \( D \subset \mathbb{R}^2 \) given by the equation

\[
f(x, y) = k, \quad (x, y) \in D, \quad k \in R.
\]

Remark: Contour curves are the intersection of the graph of \( f \) with horizontal planes \( z = k \).

Remark: Level curves are the vertical translation of contour curves to the function domain.

Example
Find and draw few level curves and contour curves for the function \( f(x, y) = x^2 + y^2 \).

Solution:
The level curves are solutions of the equation \( x^2 + y^2 = k \) with \( k \geq 0 \). These curves are circles of radius \( \sqrt{k} \) in \( D = \mathbb{R}^2 \).

The contour curves are the circles \( \{(x, y, z) : x^2 + y^2 = k, \ z = k\} \).
Example
Find the maximum domain, range of, and graph the function
\[ f(x, y) = \frac{1}{1 + x^2 + y^2}. \]

Solution:
Since the denominator never vanishes, hence \( D = \mathbb{R}^2 \).
Since \( 0 < \frac{1}{1 + x^2 + y^2} \leq 1 \), the range of \( f \) is \( R = (0, 1] \).
The contour curves are circles on horizontal planes in \( (0, 1] \).  

Example
Given the topographic map in the figure, which way do you choose to the summit?

Solution:
From the east side.
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Scalar functions of three variables

**Definition**

The *graph* of a scalar function of three variables, \( f : D \subset \mathbb{R}^3 \rightarrow R \subset \mathbb{R} \), is the set of points in \( \mathbb{R}^4 \) of the form \((x, y, z, f(x, y, z))\) for every \((x, y, z) \in D\).

**Remark:**

The graph a function \( f : D \subset \mathbb{R}^3 \rightarrow \mathbb{R} \) requires four space dimensions. We cannot picture such graph.

**Definition**

The *level surfaces* of a function \( f : D \subset \mathbb{R}^3 \rightarrow R \subset \mathbb{R} \) are the surfaces in the domain \( D \subset \mathbb{R}^3 \) of \( f \) solutions of the equation \( f(x, y, z) = k \), where \( k \in R \) is a constant in the range of \( f \).
Scalar functions of three variables

Example
Draw one level surface of the function $f : D \subset \mathbb{R}^3 \rightarrow R \subset \mathbb{R}$

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}.$$ 

Solution:
The domain of $f$ is $D = \mathbb{R}^3$, the range is $R = (0, \infty)$. For $k > 0$ the level surfaces $f(x, y, z) = k$ are

$$x^2 + y^2 + z^2 = \frac{1}{k},$$

spheres radius $R = \frac{1}{\sqrt{k}}$. \triangleleft