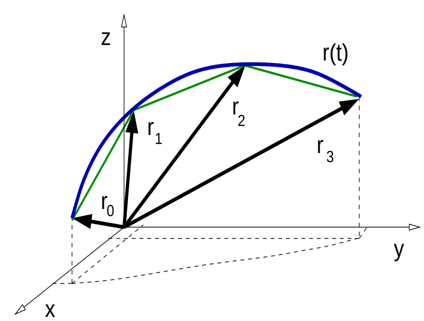
The length of a curve in space (Sect. 13.3)

- The length of a curve in space.
- The length function.
- Parametrizations of a curve.
- The length parametrization of a curve.

The length of a curve in space

Definition

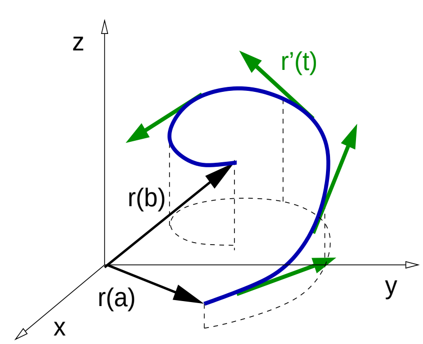
The *length* or *arc length* of a curve in the plane or in space is the limit of the polygonal line length, as the polygonal line approximates the original curve.



Theorem

The length of a continuously differentiable curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$*, with n=2,3, is the number*

$$\ell_{ba} = \int_a^b \left|\mathbf{r}'(t)\right| \, dt.$$



The length of a curve in space

Recall: The length of $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is $\ell_{ba} = \int_a^b \big|\mathbf{r}'(t)\big|\, dt$.

- If the curve $\mathbf{r}$ is the path traveled by a particle in space, then $\mathbf{r}' = \mathbf{v}$ is the velocity of the particle.
- The length is the integral in time of the particle speed $|\mathbf{v}(t)|$.
- Therefore, the length of the curve is the distance traveled by the particle.
- In Cartesian coordinates the functions $\mathbf{r}$ and $\mathbf{r}'$ are given by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \qquad \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Therefore the curve length is given by the expression

$$\ell_{ba} = \int_a^b \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2 + \big[z'(t)\big]^2}\, dt.$$

The length of a curve in a plane

Example

Find the length of the curve $\mathbf{r}(t) = \langle r_0 \cos(t), r_0 \sin(t)\rangle$, for $t \in [\pi/4, 3\pi/4]$, and $r_0 > 0$.

Solution: Compute $\mathbf{r}'(t) = \langle -r_0 \sin(t), r_0 \cos(t)\rangle$. The length of the curve is given by the formula

$$\ell = \int_{\pi/4}^{3\pi/4} \sqrt{\big[-r_0 \sin(t)\big]^2 + \big[r_0 \cos(t)\big]^2}\, dt$$

$$\ell = \int_{\pi/4}^{3\pi/4} \sqrt{r_0^2\big(\big[-\sin(t)\big]^2 + \big[\cos(t)\big]^2\big)}\, dt = \int_{\pi/4}^{3\pi/4} r_0\, dt.$$

Hence, $\ell = \frac{\pi}{2} r_0$. (The length of quarter circle of radius r_0.) $\triangleleft$

The length of a curve in a plane.

Example

Find the length of the spiral $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$, for $t \in [0, t_0]$.

Solution: The derivative vector is

$$\mathbf{r}'(t) = \Big\langle \big[-t\sin(t) + \cos(t)\big], \big[t\cos(t) + \sin(t)\big] \Big\rangle,$$

$$\begin{aligned} |\mathbf{r}'(t)|^2 &= \big[t^2\sin^2(t) + \cos^2(t) - 2t\sin(t)\cos(t)\big] \\ &\quad + \big[t^2\cos^2(t) + \sin^2(t) + 2t\sin(t)\cos(t)\big] \end{aligned}$$

We obtain $|\mathbf{r}'(t)|^2 = t^2 + 1$. The curve length is given by

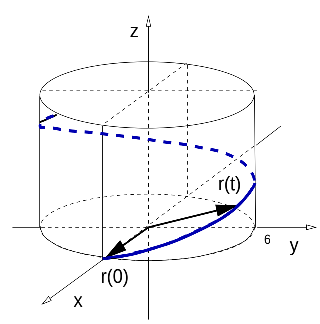
$$\ell(t_0) = \int_0^{t_0} \sqrt{1 + t^2}\, dt = \ln\big(t + \sqrt{1 + t^2}\big)\Big|_0^{t_0}.$$

We conclude that $\ell(t_0) = \ln\big(t_0 + \sqrt{1 + t_0^2}\big)$. $\triangleleft$

The length of a curve in space.

Example

Find the length of the curve $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t\rangle$, for $t \in [0, \pi]$.



Solution: The derivative vector is

$$\mathbf{r}'(t) = \langle -12\sin(2t), 12\cos(2t), 5\rangle,$$

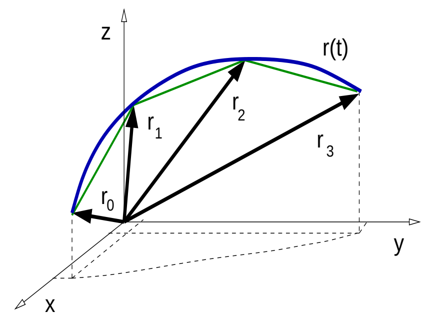
$$|\mathbf{r}'(t)|^2 = 144\big[\sin^2(2t) + \cos^2(2t)\big] + 25 = 169 = (13)^2.$$

The curve length is

$$\ell = \int_0^{\pi} 13\, dt = 13\, t\Big|_0^{\pi} \quad \Rightarrow \quad \ell = 13\pi. \qquad \triangleleft$$

The length of a curve in space.

Idea of the Proof: The curve length is the limit of the polygonal line length, as the polygonal line approximates the original curve.



$$\ell_N = \sum_{n=0}^{N-1} |\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)|, \qquad \{a = t_0, t_1, \cdots, t_{N-1}, t_N = b\},$$

$$\ell_N \simeq \sum_{n=0}^{N-1} |\mathbf{r}'(t_n)|\,(t_{n+1} - t_n) \xrightarrow{N\to\infty} \int_a^b |\mathbf{r}'(t)|\,dt.$$

□

The arc length of a curve in space (Sect. 13.3)

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The length function

Definition

The *length function* of a continuously differentiable vector function $\mathbf{r}$ is given by

$$\ell(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)|\, d\tau.$$

Remarks:

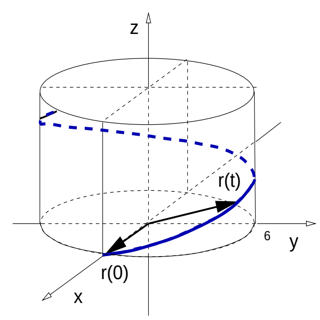
(a) The value $\ell(t)$ of the length function is the length along the curve $\mathbf{r}$ from t_0 to t.

(b) If the function $\mathbf{r}$ is the position of a moving particle as function of time, then the value $\ell(t)$ is the distance traveled by the particle from the time t_0 to t.

The length function

Example

Find the arc length function for the curve $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t\rangle$, starting at $t = 1$.



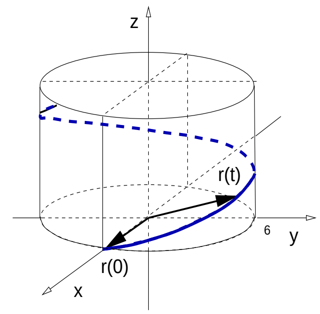
Solution: We have found that $|\mathbf{r}'(t)| = 13$. Therefore,

$$\ell(t) = \int_1^t 13\, d\tau \quad \Rightarrow \quad \ell(t) = 13\,(t-1). \qquad \triangleleft$$

The length function

Example

Given the position function in time $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t \rangle$, find the position vector $\mathbf{r}(t_0)$ located at a length $\ell_0 = 4$ from the initial position $\mathbf{r}(0)$.



Solution: We have found that the length function for $\mathbf{r}$ starting at $t = 1$ is $\hat{\ell}(t) = 13\,(t - 1)$.

It is simple to see that the length function for $\mathbf{r}$ starting at $t = 0$ is $\ell(t) = 13\,t$.

Since $t = \ell/13$, the time at $\ell_0 = 4$ is $t_0 = 4/13$.

Therefore, the position vector at $\ell_0 = 4$ is given by

$$\mathbf{r}(t_0) = \langle 6\cos(8/13), 6\sin(8/13), 20/13 \rangle. \qquad \triangleleft$$

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Parametrizations of a curve

Remark:
A curve in space can be represented by different vector functions.

Example
The unit circle in $\mathbb{R}^2$ is the curve represented by the following vector functions:

- $\mathbf{r}_1(t) = \langle \cos(t), \sin(t) \rangle$;
- $\mathbf{r}_2(t) = \langle \cos(5t), \sin(5t) \rangle$;
- $\mathbf{r}_3(t) = \langle \cos(e^t), \sin(e^t) \rangle$.

Remark:
The curve in space is the same for all three functions above. The vector $\mathbf{r}$ moves along the curve at different speeds for the different parametrizations.

Parametrizations of a curve

Remarks:

- If the vector function $\mathbf{r}$ represents the position in space of a moving particle, then there is a preferred parameter to describe the motion. The time t.
- Another preferred parameter useful to describe a moving particle is the distance traveled by the particle. The length ℓ.
- The latter parameter is defined for every curve, either the curve represents motion or not.
- A common problem when describing motion is the following: Given a vector function parametrized by the time t, switch the curve parameter to the curve length ℓ.
- This is called the curve length parametrization.

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The length parametrization of a curve

Problem:
Given vector function $\mathbf{r}$ in terms of a parameter t, find the arc length parametrization of that curve.

Solution:

(a) With the function values $\mathbf{r}(t)$ compute the arc length function $\ell(t)$, starting at some $t = t_0$.

(b) Invert the function values $\ell(t)$ to find the function values $t(\ell)$.

(c) Example: If $\ell(t) = 3e^{t/2}$, then $t(\ell) = 2\ln(\ell/3)$.

(d) Compute the composition function $\hat{\mathbf{r}}(\ell) = \mathbf{r}(t(\ell))$. That is, replace t by $t(\ell)$ in the function values $\mathbf{r}(t)$.

Remark: The function values $\hat{\mathbf{r}}(\ell)$ are the parametrization of the function values $\mathbf{r}(t)$ using the curve length as the new parameter.

The length parametrization of a curve

Example

Find the curve length parametrization of the vector function $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ starting at $t = 1$.

Solution: First find the derivative function:

$$\mathbf{r}'(t) = \langle -4\sin(t), 4\cos(t), 3\rangle.$$

Hence, $|\mathbf{r}'(t)|^2 = 4^2\sin^2(t) + 4^2\cos^2(t) + 3^2 = 16 + 9 = 5^2$.

Find the arc length function: $\ell(t) = \displaystyle\int_1^t 5\,d\tau \Rightarrow \ell(t) = 5(t-1)$.

Invert the equation above: $t = \dfrac{\ell}{5} + 1$, that is, $t = \dfrac{(\ell+5)}{5}$.

So, $\hat{\mathbf{r}}(\ell) = \Big\langle 4\cos\Big[\dfrac{(\ell+5)}{5}\Big], 4\sin\Big[\dfrac{(\ell+5)}{5}\Big], \dfrac{3(\ell+5)}{5}\Big\rangle.$ $\triangleleft$

The length parametrization of a curve

Theorem

If the continuously differentiable curve $\mathbf{r}$ *has length parametrization values* $\hat{\mathbf{r}}(\ell)$*, then* $\mathbf{u}(\ell) = \dfrac{d\hat{\mathbf{r}}}{d\ell}$ *is a unit vector tangent to the curve.*

Proof:

Given the function values $\mathbf{r}(t)$, let $\hat{\mathbf{r}}(\ell)$ be the reparametrization of $\mathbf{r}$ with the curve length function $\ell(t) = \displaystyle\int_{t_0}^t |\mathbf{r}'(\tau)|\,d\tau$.

Notice that $\dfrac{d\ell}{dt} = |\mathbf{r}'(t)|$ and $\dfrac{dt}{d\ell} = \dfrac{1}{|\mathbf{r}'(t)|}$.

Therefore, $\mathbf{u}(\ell) = \dfrac{d\hat{\mathbf{r}}(\ell)}{d\ell} = \dfrac{d\mathbf{r}(t)}{dt}\,\dfrac{dt}{d\ell} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

We conclude that $|\mathbf{u}(\ell)| = 1$. $\square$

The length parametrization of a curve

Example

Find a unit vector tangent to the curve given by
$\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ for $t \geqslant 0$.

Solution: Reparametrize the curve using the arc length.
Recall: $|\mathbf{r}'(t)| = 5$, and $\ell(t) = 5t$, so $t = \ell/5$. We get

$$\hat{\mathbf{r}}(\ell) = \langle 4\cos(\ell/5), 4\sin(\ell/5), 3\ell/5\rangle.$$

Therefore, a unit tangent vector is

$$\mathbf{u}(\ell) = \frac{d\hat{\mathbf{r}}}{d\ell} \quad \Rightarrow \quad \mathbf{u}(\ell) = \Big\langle -\frac{4}{5}\sin(\ell/5), \frac{4}{5}\cos(\ell/5), \frac{3}{5}\Big\rangle. \quad \lhd$$

We can verify that this is a unit vector, since

$$|\mathbf{u}(\ell)|^2 = \Big(\frac{4}{5}\Big)^2\big[\sin^2(\ell/5) + \cos^2(\ell/5)\big] + \Big(\frac{3}{5}\Big)^2 \quad \Rightarrow \quad |\mathbf{u}(\ell)| = 1.$$