Integration and projectile motion (Sect. 13.2)

- Integration of vector functions.
- Application: Projectile motion.
  - Equations of a projectile motion.
  - Range, Height, Flight Time.

Integration of vector functions

Definition

An *antiderivative* of a vector function $\mathbf{v}$ is any vector valued function $\mathbf{V}$ such that $\mathbf{V}' = \mathbf{v}$.

Remark: Antiderivatives are also called *indefinite integrals*, or *primitives*, they are denoted as $\int \mathbf{v}(t)\,dt$, that is,

$$\int \mathbf{v}(t)\,dt = \mathbf{V}(t) + \mathbf{C},$$

where $\mathbf{C}$ is a constant vector in Cartesian coordinates.

Example

Verify that $\mathbf{V} = \langle(-\cos(3t)/3 + 1), (\sin(t) - 2), (e^{2t}/2 + 2)\rangle$ is an antiderivative of $\mathbf{v} = \langle \sin(3t), \cos(t), e^{2t}\rangle$.

Solution: $\mathbf{V}' = \langle(-\cos(3t)/3 + 1)', (\sin(t) - 2)', (e^{2t}/2 + 2)'\rangle = \mathbf{v}$.

Integrals of vector functions.

Example

Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$.

Solution: The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,$$

with $\mathbf{C} = \langle c_x, c_y, c_z \rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:

$$\langle 1, 1, 1 \rangle = \mathbf{r}(0) = \langle 0, 0, -1 \rangle + \langle c_x, c_y, c_z \rangle,$$

that is, $c_x = 1$, $c_y = 1$, $c_z = 2$.

The position function is $\mathbf{r}(t) = \langle t^2 + 1, \sin(t) + 1, -\cos(t) + 2 \rangle$. $\triangleleft$

Integrals of vector functions.

Example

Find the position function of a particle with acceleration $\mathbf{a}(t) = \langle 0, 0, -10 \rangle$ having an initial velocity $\mathbf{v}(0) = \langle 0, 1, 1 \rangle$ and initial position $\mathbf{r}(0) = \langle 1, 0, 1 \rangle$.

Solution: The velocity is the antiderivative of the acceleration:

$$\mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle,$$

where $\mathbf{v}_0 = \langle v_{0x}, v_{0y}, v_{0z} \rangle$ is fixed by the initial condition.

$$\mathbf{v}(0) = \langle 0, 1, 1 \rangle = \langle v_{0x}, v_{0y}, v_{0z} \rangle$$

The velocity function is $\mathbf{v}(t) = \langle 0, 1, (-10t + 1) \rangle$.

The position is $\mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z}) \rangle$, and

$$\mathbf{r}(0) = \langle 1, 0, 1 \rangle = \langle r_{0x}, r_{0y}, r_{0z} \rangle,$$

The obtain that $\mathbf{r}(t) = \langle 1, t, (-5t^2 + t + 1) \rangle$. $\triangleleft$

Integrals of vector functions.

Definition

The *definite integral* of an integrable vector function $\mathbf{r}(t) = \langle \mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t) \rangle$ on the interval $[a, b]$ is given by

$$\int_a^b \mathbf{r}(t)dt = \Big\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \Big\rangle.$$

Example

Compute $\displaystyle\int_0^\pi \mathbf{r}(t)\,dt$ for the function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution: We compute an antiderivative and we evaluate the result,

$$\mathbf{I} = \int_0^\pi \mathbf{r}(t)\,dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle dt.$$

Integrals of vector functions.

Example

Compute $\displaystyle\int_0^\pi \mathbf{r}(t)\,dt$ for the function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:

$$\mathbf{I} = \int_0^\pi \mathbf{r}(t)\,dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle dt.$$

$$\mathbf{I} = \Big\langle \int_0^\pi \cos(t)dt, \int_0^\pi \sin(t)dt, \int_0^\pi tdt \Big\rangle,$$

$$\mathbf{I} = \Big\langle \sin(t)\Big|_0^\pi, -\cos(t)\Big|_0^\pi, \frac{t^2}{2}\Big|_0^\pi \Big\rangle$$

$$\mathbf{I} = \Big\langle 0, 2, \frac{\pi^2}{2} \Big\rangle \quad \Rightarrow \quad \int_0^\pi \mathbf{r}(t)\,dt = \Big\langle 0, 2, \frac{\pi^2}{2} \Big\rangle. \qquad \triangleleft$$

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Equations of a projectile motion

Remark: Projectile motion is the position of a point particle moving near the Earth surface subject to gravitational attraction.

Theorem

The motion of a particle with initial velocity $\mathbf{v}_0$ and position $\mathbf{r}_0$ subject to an acceleration $\mathbf{a} = -g\mathbf{k}$, where g is a constant, is

$$\mathbf{r}(t) = -\frac{g}{2}t^2\mathbf{k} + \mathbf{v}_0 t + \mathbf{r}_0.$$

Remarks:

(a) The equation above in vector components is

$$\mathbf{r}(t) = \Big\langle (v_{0x}t + r_{0x}), (v_{0y}t + r_{0y}), \Big(-\frac{g}{2}\,t^2 + v_{0z}t + r_{0z}\Big)\Big\rangle,$$

where $\mathbf{v}_0 = \langle v_{0x}, v_{0y}, v_{0z}\rangle$ and $\mathbf{r}_0 = \langle r_{0x}, r_{0y}, r_{0z}\rangle$.

(b) The motion occurs in a plane. We describe it with vectors in the plane $\mathbb{R}^2$. We use the coordinates x, y, only.

Equations of a projectile motion

Remark: Same Theorem, written in x, y coordinates in $\mathbb{R}^2$.

Theorem

The motion of a particle with initial velocity $\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$ *and position* $\mathbf{r}_0 = r_{0x}\mathbf{i} + r_{0y}\mathbf{j}$ *subject to the acceleration* $\mathbf{a} = -g\mathbf{j}$ *, where* g *is a constant, is*

$$\mathbf{r}(t) = -\frac{g}{2}\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0,$$

equivalently, $\mathbf{r}(t) = (v_{0x}t + r_{0x})\mathbf{i} + \Big(-\frac{g}{2}t^2 + v_{0y}t + r_{0y}\Big)\mathbf{j}.$

Proof: Since $\mathbf{r}''(t) = -g\mathbf{j}$, then $\mathbf{r}'(t) = c_x\mathbf{i} + (-gt + c_y)\mathbf{j}$.

$$\mathbf{r}'(0) = v_{0x}\mathbf{i} + v_{0y}\mathbf{j} = c_x\mathbf{i} + c_y\mathbf{j} \;\Rightarrow\; \mathbf{r}'(t) = v_{0x}\mathbf{i} + (-gt + v_{0y})\mathbf{j}\,.$$

One more integration, $\mathbf{r}(t) = \big(d_x + v_{0x}t\big)\mathbf{i} + \big(d_y + v_{0y}t - \frac{g}{2}t^2\big)\mathbf{j}$.

The initial condition $\mathbf{r}(0) = r_{0x}\mathbf{i} + r_{0y}\mathbf{j} = d_x\mathbf{i} + d_y\mathbf{j}$,

implies that $\mathbf{r}(t) = (v_{0x}t + r_{0x})\mathbf{i} + \Big(-\frac{g}{2}t^2 + v_{0y}t + r_{0y}\Big)\mathbf{j}.$ □

Equations of a projectile motion

Example

Find the position function and the trajectory of a projectile with initial speed $|\mathbf{v}_0| = 4$ m/s, launched from the coordinate system origin with an elevation angle of $\theta = \pi/3$.

Solution: The projectile acceleration is $\mathbf{a} = -g\mathbf{j}$, with $g = 10$ m/s. Therefore, $\mathbf{v}(t) = (-10t + v_{0y})\mathbf{j} + v_{0x}\mathbf{i}$, where

$$v_{0y} = |\mathbf{v}_0|\sin(\theta) = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}, \quad v_{0x} = |\mathbf{v}_0|\cos(\theta) = 4\,\frac{1}{2} = 2.$$

Since $\mathbf{v}(t) = (-10t + 2\sqrt{3})\mathbf{j} + 2\mathbf{i}$ and $\mathbf{r}_0 = \mathbf{0}$, then

$$\mathbf{r}(t) = (-5t^2 + 2\sqrt{3}t)\mathbf{j} + 2t\mathbf{i}.$$

Since $y(t) = -5t^2 + 2\sqrt{3}t$ and $x(t) = 2t$, the trajectory is

$$y(x) = -5\left(\frac{x^2}{4}\right) + 2\sqrt{3}\,\frac{x}{2} \quad\Rightarrow\quad y(x) = -\frac{5}{4}x^2 + \sqrt{3}\,x.$$ ◁

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Range, Height, Flight Time

Theorem

The the range x_r, height y_h, and the fight time t_r of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ *are*

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \qquad y_h = \frac{(v_{0y})^2}{2g}, \qquad t_r = \frac{2v_{0y}}{g}.$$

Remark: Since the initial speed $|\mathbf{v}_0|$ and the elevation angle θ determine v_{0y} and v_{0x} by the equations

$$v_{0y} = |\mathbf{v}_0| \sin(\theta), \qquad v_{0x} = |\mathbf{v}_0| \cos(\theta),$$

then holds

$$x_r = \frac{|\mathbf{v}_0|^2 \sin(2\theta)}{g}, \quad y_h = \frac{|\mathbf{v}_0|^2 \sin^2(\theta)}{2g}, \quad t_r = \frac{2|\mathbf{v}_0| \sin(\theta)}{g}.$$

Range, Height, Flight Time

Theorem

The the range x_r, height y_h, and the fight time t_r of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \qquad y_h = \frac{(v_{0y})^2}{2g}, \qquad t_r = \frac{2v_{0y}}{g}.$$

Proof: Since $\mathbf{r}_0 = \mathbf{0}$, the expression for the projectile position function $\mathbf{r}(t) = y(t)\,\mathbf{j} + x(t)\,\mathbf{i}$ is

$$y(t) = -\frac{g}{2}\,t^2 + v_{0y}\,t, \qquad x(t) = v_{0x}\,t.$$

Using $t = x/v_{0x}$ we get the trajectory

$$y(x) = -\frac{g}{2v_{0x}^2}\,x^2 + \frac{v_{0y}}{v_{0x}}\,x.$$

Range, Height, Flight Time

Theorem

The the range x_r, height y_h, and the fight time t_r of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \qquad y_h = \frac{(v_{0y})^2}{2g}, \qquad t_r = \frac{2v_{0y}}{g}.$$

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2}\,x^2 + \frac{v_{0y}}{v_{0x}}\,x$. The range is given by the condition $y(x_r) = 0$ and $x_r \neq 0$, that is,

$$-\frac{g}{2v_{0x}}\,x_r + v_{0y} = 0 \quad\Rightarrow\quad x_r = \frac{2v_{0x}v_{0y}}{g}.$$

The maximum height occurs where $y'(x) = 0$, that is,

$$-\frac{g}{v_{0x}^2}x_h + \frac{v_{0y}}{v_{0x}} = 0 \quad\Rightarrow\quad x_h = \frac{v_{0x}v_{0y}}{g} \quad\Rightarrow\quad x_h = \frac{x_r}{2}.$$

Range, Height, Flight Time

Theorem

The the range x_r, *height* y_h, *and the fight time* t_r *of a projectile launched from the origin with initial velocity* $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ *are*

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \qquad y_h = \frac{(v_{0y})^2}{2g}, \qquad t_r = \frac{2v_{0y}}{g}.$$

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2}x^2 + \frac{v_{0y}}{v_{0x}}x$, and $x_h = \frac{v_{0x}v_{0y}}{g}$.

Then, the maximum height $y_h = y(x_h)$ is

$$y_h = -\frac{g}{2v_{0x}^2}\frac{v_{0x}^2v_{0y}^2}{g^2} + \frac{v_{0y}}{v_{0x}}\frac{v_{0x}v_{0y}}{g} = -\frac{v_{0y}^2}{2g} + \frac{v_{0y}^2}{g} \quad \Rightarrow \quad y_h = \frac{v_{0y}^2}{2g}.$$

Recalling that $x(t) = v_{0x}t$, then the flight time t_r is

$$t_r = \frac{x_r}{v_{0x}}2 \quad \Rightarrow \quad t_r = \frac{2v_{0y}}{g}.$$

□

Range, Height, Flight Time

Example

Find the range, height and flight time of the projectile with initial velocity $\mathbf{v}_0 = 3\mathbf{j} + \mathbf{i}$.

Solution: We could use the formulas from the Theorem. However, we compute them following the Theorem proof.

From $\mathbf{a} = -10\,\mathbf{j}$ we get the projectile position function,

$$y(t) = -5t^2 + 3t, \qquad x(t) = t.$$

The trajectory is $y(x) = -5x^2 + 3x$. The range is

$$y(x_r) = 0 = -5x_r + 3 \quad \Rightarrow \quad x_r = \frac{3}{5}.$$

The height is $y_h = y\Big(\frac{x_r}{2}\Big)$, so, $y_h = -5\,\frac{3^2}{10^2} + 3\,\frac{3}{10}$, so $y_h = \frac{9}{20}$.

The time flight is $t_r = \frac{x_r}{v_{0x}}$, that is, $t_r = \frac{3}{5}$. ◁