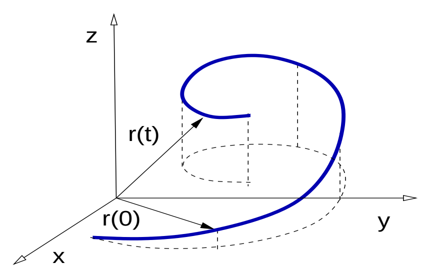
Vector functions (Sect. 13.1)

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Motion on a sphere.

Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$

Definition

A *vector function* is function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, and the function *domain* is the interval $I \subset \mathbb{R}$.



Remarks:

(a) Motion in space motivates to define vector functions.

(b) Given Cartesian coordinates in $\mathbb{R}^3$, the values of a vector function can be written in components as follows:

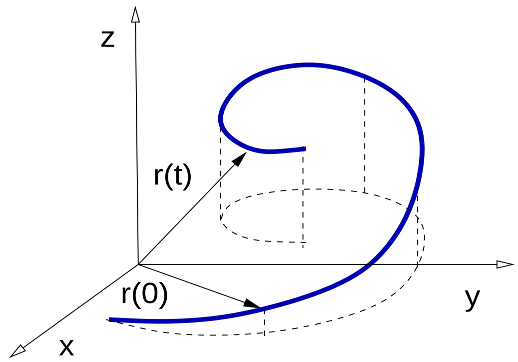
$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \qquad t \in I,$$

where $x(t)$, $y(t)$, and $z(t)$ are the values of three scalar functions.

Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$

Remarks:

- There is a natural association between a curve in $\mathbb{R}^n$ and the vector function values $\mathbf{r}(t)$.



- The curve is determined by the terminal points of the vector function values $\mathbf{r}(t)$.
- The independent variable t is called the parameter of the curve.

Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$

Example

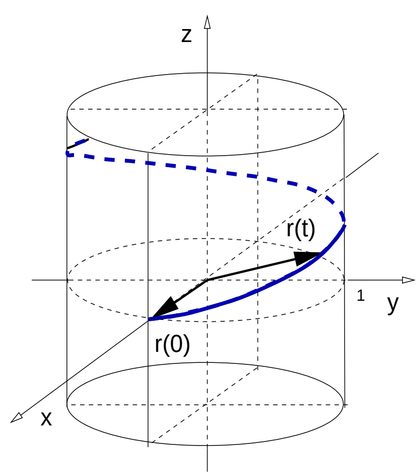
Graph the vector function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:

The curve given by $\mathbf{r}(t)$ lies on a vertical cylinder with radius one, since

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1.$$

The $z(t)$ coordinate of the curve increases with t, so the terminal point $\mathbf{r}(t)$ moves up on the cylinder surface when t increases. $\triangleleft$



Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$

Example

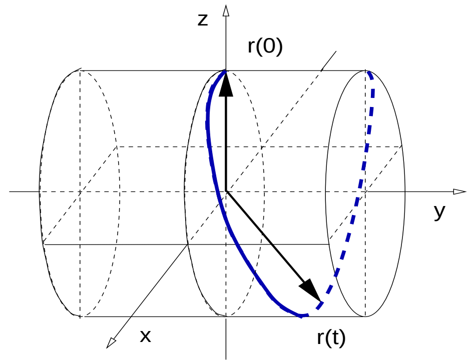
Graph the vector function $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$.

Solution:

The curve given by $\mathbf{r}(t)$ lies on a horizontal cylinder with radius one, since

$$x^2 + z^2 = \sin^2(t) + \cos^2(t) = 1.$$

The $y(t)$ coordinate of the curve increases with t, so the terminal point $\mathbf{r}(t)$ moves to the right on the cylinder surface when t increases. $\triangleleft$



Vector functions (Sect. 13.1)

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
- **Limits and continuity of vector functions.**
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Limits and continuity of vector functions

Definition

The vector function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, has a *limit* given by the vector $\mathbf{L}$ when t approaches t_0, denoted as $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{L}$, iff:

For every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$0 < |t - t_0| < \delta \quad \Rightarrow \quad |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

Remark:

- The limit of $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ as $t \to t_0$ is the limit of its components $x(t)$, $y(t)$, $z(t)$ in Cartesian coordinates.
- That is: $\lim_{t\to t_0} \mathbf{r}(t) = \Big\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\Big\rangle$.

$\lim_{t\to t_0} \mathbf{r}(t) = \Big\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\Big\rangle$

Example

Given $\mathbf{r}(t) = \langle \cos(t), \sin(t)/t, t^2 + 2\rangle$, compute $\lim_{t\to 0} \mathbf{r}(t)$.

Solution:

Notice that the vector function $\mathbf{r}$ is not defined at $t = 0$, however its limit at $t = 0$ exists. Indeed,

$$\lim_{t\to 0} \mathbf{r}(t) = \lim_{t\to 0} \Big\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2\Big\rangle$$

$$\lim_{t\to 0} \mathbf{r}(t) = \Big\langle \lim_{t\to 0} \cos(t), \lim_{t\to 0} \frac{\sin(t)}{t}, \lim_{t\to 0} (t^2 + 2)\Big\rangle$$

$$\lim_{t\to 0} \mathbf{r}(t) = \langle 1, 1, 2\rangle.$$

We conclude that $\lim_{t\to 0} \mathbf{r}(t) = \langle 1, 1, 2\rangle$. $\lhd$

Limits and continuity of vector functions.

Definition

A vector function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, is *continuous at* $t = t_0 \in I$ iff holds $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function $\mathbf{r} : I \to \mathbb{R}^n$ is *continuous* if it is continuous at every t in its domain interval I.

Remark: A vector function with Cartesian components $\mathbf{r} = \langle x, y, z \rangle$ is continuous iff each component is continuous.

Example

The function $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$ is continuous for $t \in \mathbb{R}$. $\triangleleft$

Remark: Having the idea of limit, one can introduce the idea of a derivative of a vector valued function.

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Derivatives and motion

Definition

The vector function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, is *differentiable at* $t = t_0$, denoted as $\mathbf{r}'(t)$ or $\dfrac{d\mathbf{r}}{dt}$, iff the following limit exists,

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

Remarks:

- A vector function $\mathbf{r} : I \to \mathbb{R}^n$ is *differentiable* if it is differentiable for each $t \in I$.
- If a vector function with values $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ in Cartesian components is differentiable, then

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Derivatives and motion.

Theorem

If a vector function with values $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ *in Cartesian components is differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Proof:

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

$$= \lim_{h\to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \right\rangle$$

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle. \qquad \square$$

Derivatives and motion.

Example

Find the derivative of the vector function
$\mathbf{r}(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle$.

Solution: We differentiate each component of $\mathbf{r}$, that is,

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), (2t + 3) \rangle. \qquad \triangleleft$$

Example

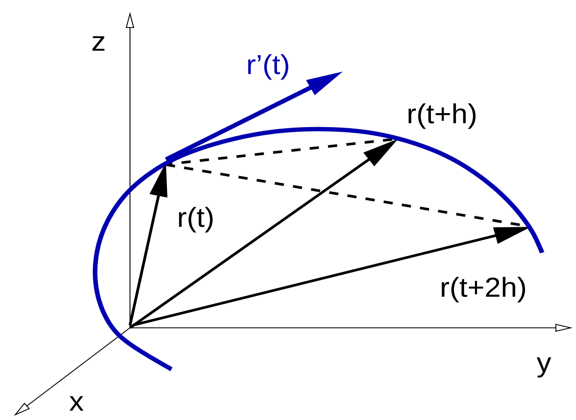
Find the derivative of the vector function $\mathbf{r}(t) = \langle \cos(2t), e^{3t}, 1/t \rangle$.

Solution: We differentiate each component of $\mathbf{r}$, that is,

$$\mathbf{r}'(t) = \langle -2\sin(2t), 3e^{3t}, -1/t^2 \rangle. \qquad \triangleleft$$

Geometrical property of the derivative

Remark: The vector $\mathbf{r}'(t)$ is tangent to the curve given by the vector function $\mathbf{r}$ at the end point of $\mathbf{r}(t)$.



Remark: If $\mathbf{r}(t)$ represents the vector position of a particle, then:

- The derivative of the position function is the velocity function, $\mathbf{v}(t) = \mathbf{r}'(t)$. The speed is $|\mathbf{v}(t)|$.
- The derivative of the velocity function is the acceleration function, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Derivatives and motion.

Example

Compute the derivative of the position function $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$. Graph the curve given by $\mathbf{r}$, and explicitly show the position vector $\mathbf{r}(0)$ and velocity vector $\mathbf{v}(0)$.

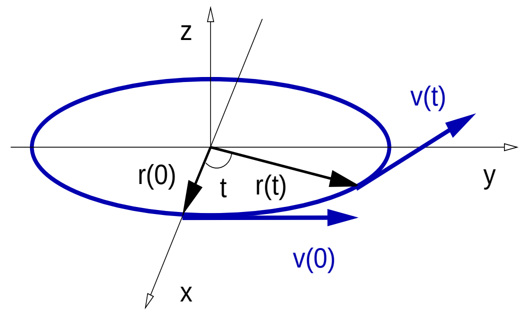
Solution:

The derivative of $\mathbf{r}$ is computed component by component,

$$\mathbf{v}(t) = \langle -\sin(t), \cos(t), 0 \rangle.$$

$\mathbf{r}(0) = \langle 1, 0, 0 \rangle$, $\mathbf{v}(0) = \langle 0, 1, 0 \rangle$.

$\triangleleft$



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Differentiation rules are the similar as for scalar functions

Theorem

If $\mathbf{v}$ *and* $\mathbf{w}$ *are differentiable vector functions, then holds:*

- $[\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t)$, *(addition);*
- $[c\mathbf{v}(t)]' = c\mathbf{v}'(t)$, *(product rule);*
- $[\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t)$, *(chain rule);*
- $[f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)$, *(product rule);*
- $[\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)$, *(dot product);*
- $[\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t)$, *(cross product).*

Higher derivatives can also be computed.

Definition

The *m-derivative* of a vector function $\mathbf{r}$ is denoted as $\mathbf{r}^{(m)}$ and is given by the expression $\mathbf{r}^{(m)} = [\mathbf{r}^{(m-1)}]'$.

Example

Compute the third derivative of $\mathbf{r}(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle$.

Solution:

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 2t + 2 \rangle,$$

$$\mathbf{r}^{(2)}(t) = \left[\mathbf{r}'\right]'(t) = \langle -\cos(t), -\sin(t), 2 \rangle,$$

$$\mathbf{r}^{(3)}(t) = \left[\mathbf{r}^{(2)}\right]'(t) = \langle \sin(t), -\cos(t), 0 \rangle.$$

◁

Recall: If $\mathbf{r}(t)$ is the position of a particle, then $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity and $\mathbf{a}(t) = \mathbf{r}^{(2)}(t)$ is the acceleration of the particle.

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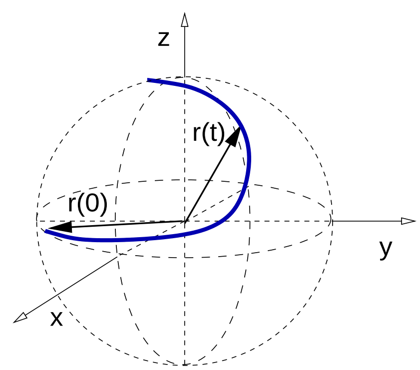
Motion in a sphere

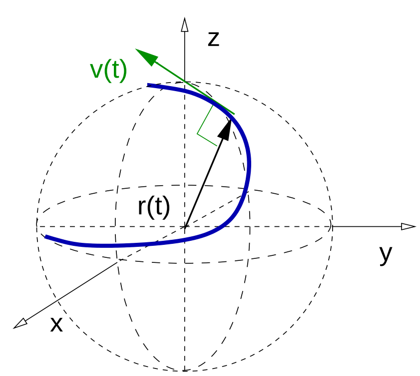
Remark: A particle with position function $\mathbf{r}$ moves on the surface of a sphere iff the vector function $\mathbf{r}$ has constant magnitude, that is, $|\mathbf{r}(t)| = r_0$ for every t in the function domain.



Theorem

If a differentiable vector function $\mathbf{r} : I \to \mathbb{R}^3$ has constant length, then for all $t \in I$ holds

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.$$



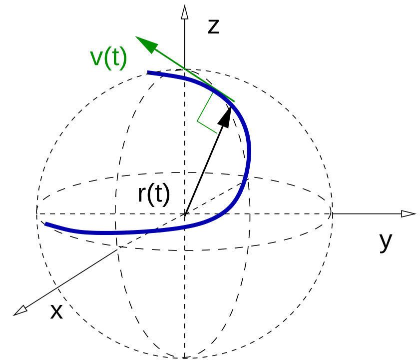
Remark: A motion on a sphere satisfies that $\mathbf{r} \perp \mathbf{v}$.

Motion in a sphere

Theorem

If a differentiable vector function $\mathbf{r} : I \to \mathbb{R}^3$ has constant length, then for all $t \in I$ holds

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.$$



Proof: Since $|\mathbf{r}(t)| = r_0$, constant for all $t \in I$,

$$\mathbf{r} \cdot \mathbf{r} = r_0^2.$$

Derivate on both sides above, use the derivative properties,

$$(\mathbf{r} \cdot \mathbf{r})' = (r_0^2)' = 0 \quad \Rightarrow \quad \mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0.$$

Since the dot product is symmetric and $\mathbf{r}' = \mathbf{v}$, we obtain that

$$\mathbf{r} \cdot \mathbf{r}' = 0 \quad \Leftrightarrow \quad \mathbf{r} \cdot \mathbf{v} = 0.$$

□

Motion in a sphere

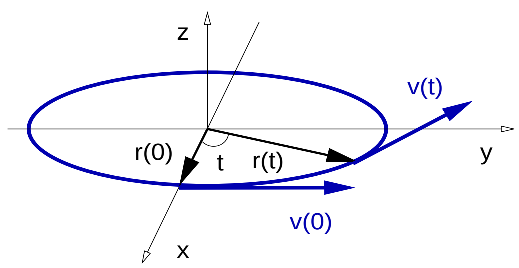
Example

Show that the position vector $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ of a particle moving in a circle is perpendicular to its velocity for $t \in \mathbb{R}$.

Solution:

We compute its velocity vector,

$$\mathbf{v}(t) = \langle -\sin(t), \cos(t), 0 \rangle.$$



Then we compute $\mathbf{r}(t) \cdot \mathbf{v}(t)$, that is,

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = -\cos(t)\sin(t) + \sin(t)\cos(t) = 0.$$

◁