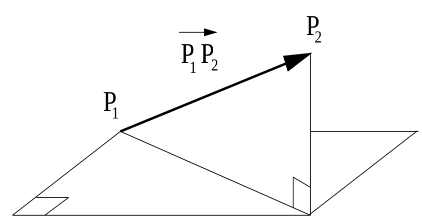
Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition

A *vector* in $\mathbb{R}^n$, with $n = 2, 3$, is an ordered pair of points in $\mathbb{R}^n$, denoted as $\overrightarrow{P_1 P_2}$, where P_1, $P_2 \in \mathbb{R}^n$. The point P_1 is called the *initial point* and P_2 is called the *terminal point*.

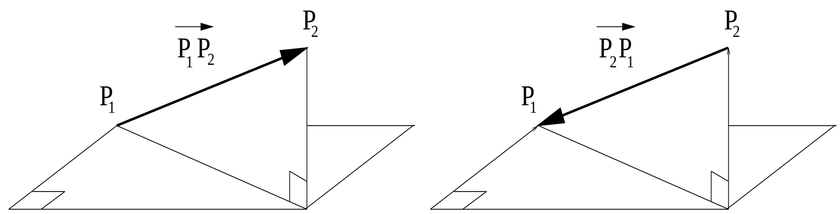


Remarks:

- A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an oriented line segment.
- A vector is drawn by an arrow pointing to the terminal point.
- A vector is denoted not only by $\overrightarrow{P_1 P_2}$ but also by an arrow over a letter, like $\vec{v}$, or by a boldface letter, like $\mathbf{v}$.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Remark: The order of the points determines the direction. For example, the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ have opposite directions.



Remark: By 1850 it was realized that different physical phenomena were described using a new concept at that time, called a vector. A vector was more than a number in the sense that it was needed more than a single number to specify it. Phenomena described using vectors included velocities, accelerations, forces, rotations, electric phenomena, magnetic phenomena, and heat transfer.

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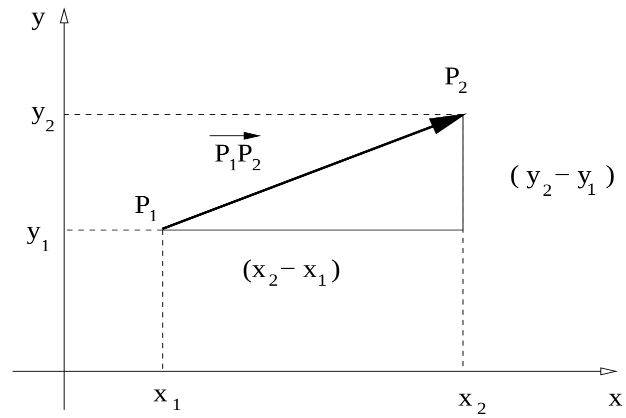
Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, the vector $\overrightarrow{P_1P_2}$ determines a unique ordered pair, called vector components,

$$\langle \overrightarrow{P_1P_2} \rangle = \langle (x_2 - x_1), (y_2 - y_1) \rangle.$$

Proof:
Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



Remark: A similar result holds for vectors in space.

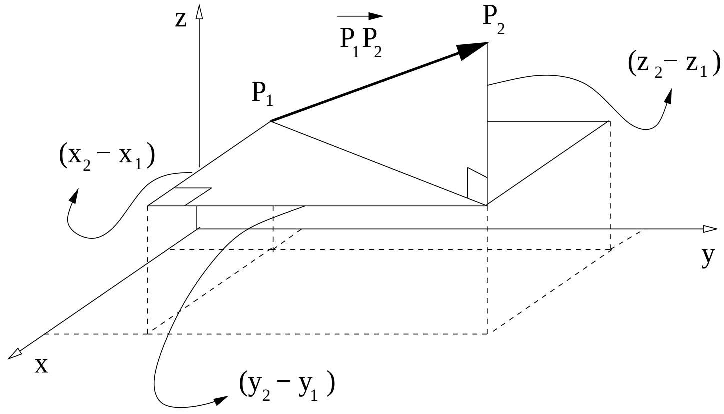
Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, the vector $\overrightarrow{P_1P_2}$ fixes a unique ordered triple, called vector components,

$$\langle \overrightarrow{P_1P_2} \rangle = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle.$$

Proof:
Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



Components of a vector in Cartesian coordinates

Example

Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

Solution:

$$\langle \overrightarrow{P_1P_2} \rangle = \langle (3-1), (1-(-2)), (2-3) \rangle \quad \Rightarrow \quad \langle \overrightarrow{P_1P_2} \rangle = \langle 2, 3, -1 \rangle.$$

Example

Find the components of a vector with initial point $P_3 = (3, 1, 4)$ and terminal point $P_4 = (5, 4, 3)$.

Solution:

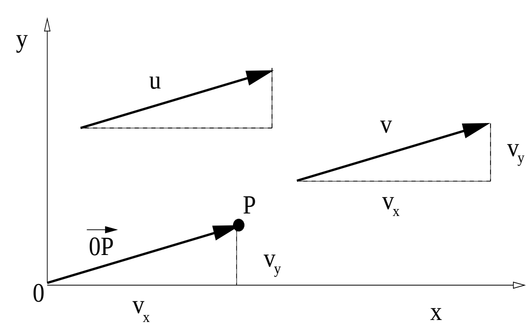
$$\langle \overrightarrow{P_3P_4} \rangle = \langle (5-3), (4-1), (3-4) \rangle \quad \Rightarrow \quad \langle \overrightarrow{P_3P_4} \rangle = \langle 2, 3, -1 \rangle.$$

Remark: $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ have the same components although they are different vectors.

Components of a vector in Cartesian coordinates

Remark:
The vector components determine a vector up to translations.

Notice that $\mathbf{u} \neq \mathbf{v} \neq \overrightarrow{0P}$, since they have different initial and terminal points. However, $\langle \mathbf{u} \rangle = \langle \mathbf{v} \rangle = \langle \overrightarrow{0P} \rangle = \langle v_x, v_y \rangle$.



Definition

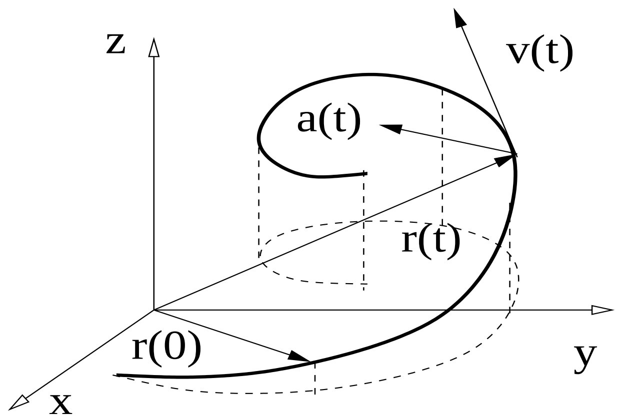
The *standard position vector* of a vector with components $\langle v_x, v_y \rangle$ is the vector $\overrightarrow{0P}$, where the point $0 = (0, 0)$ is the origin of the Cartesian coordinates and the point $P = (v_x, v_y)$.

Notation: We identify vectors with their components: $\mathbf{v} = \langle \mathbf{v} \rangle$.

Components of a vector in Cartesian coordinates

Remark: Vectors are used to describe motion of particles.

The position $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$ at the time t of a moving particle are described by vectors in space.



Vectors on a plane and in space (12.2)

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Magnitude of a vector and unit vectors.

Definition

The *magnitude* or *length* of a vector $\overrightarrow{P_1P_2}$ is the distance from the initial point to the terminal point.

- If the vector $\overrightarrow{P_1P_2}$ has components

  $$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle,$$

  then its magnitude, denoted as $|\overrightarrow{P_1P_2}|$, is given by

  $$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- If the vector $\mathbf{v}$ has components $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, then its magnitude, denoted as $|\mathbf{v}|$, is given by

  $$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Magnitude of a vector and unit vectors.

Example

Find the length of a vector with initial point $P_1 = (1, 2, 3)$ and terminal point $P_2 = (4, 3, 2)$.

Solution: First find the component of the vector $\overrightarrow{P_1P_2}$, that is,

$$\overrightarrow{P_1P_2} = \langle (4 - 1), (3 - 2), (2 - 3) \rangle \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle 3, 1, -1 \rangle.$$

Therefore, its length is

$$|\overrightarrow{P_1P_2}| = \sqrt{3^2 + 1^2 + (-1)^2} \quad \Rightarrow \quad |\overrightarrow{P_1P_2}| = \sqrt{11}.$$

Example

If the vector $\mathbf{v}$ represents the velocity of a moving particle, then its length $|\mathbf{v}|$ represents the speed of the particle. $\triangleleft$

Magnitude of a vector and unit vectors.

Definition

A vector $\mathbf{v}$ is a *unit vector* iff $\mathbf{v}$ has length one, that is, $|\mathbf{v}| = 1$.

Example

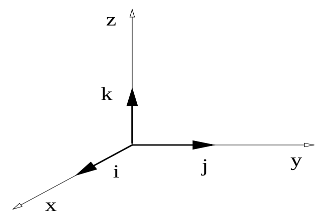
Show that $\mathbf{v} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$ is a unit vector.

Solution:

$$|\mathbf{v}| = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{14}{14}} \quad \Rightarrow \quad |\mathbf{v}| = 1.$$

Example

The unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are useful to express any other vector in $\mathbb{R}^3$.



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Addition and scalar multiplication.

Definition

Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the *vector addition*, $\mathbf{v} + \mathbf{w}$, and the *scalar multiplication*, $a\mathbf{v}$, are given by

$$\mathbf{v} + \mathbf{w} = \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle,$$
$$a\mathbf{v} = \langle av_x, av_y, av_z \rangle.$$

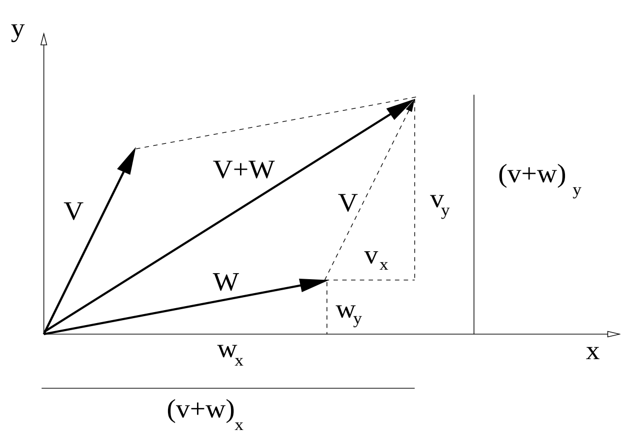
Remarks:

- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the *opposite* of vector $\mathbf{v}$.
- The difference of two vectors is the addition of one vector and the opposite of the other vector, that is, $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$. This equation in components is

$$\mathbf{v} - \mathbf{w} = \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle.$$

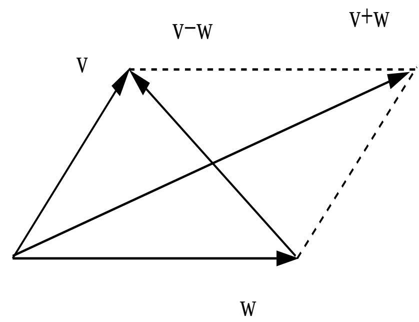
Addition and scalar multiplication.

Remark: The addition of two vectors is equivalent to the parallelogram law: The vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$ when they are in their standard position.

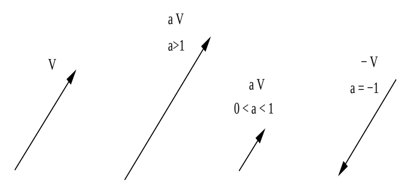


Addition and scalar multiplication.

Remark: The addition and difference of two vectors.



Remark: The scalar multiplication stretches a vector if $a > 1$ and compresses the vector if $0 < a < 1$.



Addition and scalar multiplication.

Example

Given the vectors $\mathbf{v} = \langle 2, 3\rangle$ and $\mathbf{w} = \langle -1, 2\rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$, that is,

$$\mathbf{v} + \mathbf{w} = \langle (2-1), (3+2)\rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5\rangle.$$

Therefore, its magnitude is

$$|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2} \quad \Rightarrow \quad |\mathbf{v} + \mathbf{w}| = \sqrt{26}.$$

A similar calculation can be done for $\mathbf{v} - \mathbf{w}$, that is,

$$\mathbf{v} - \mathbf{w} = \langle (2+1), (3-2)\rangle \quad \Rightarrow \quad \mathbf{v} - \mathbf{w} = \langle 3, 1\rangle.$$

Therefore, its magnitude is

$$|\mathbf{v} - \mathbf{w}| = \sqrt{3^2 + 1^2} \quad \Rightarrow \quad |\mathbf{v} - \mathbf{w}| = \sqrt{10}.$$

Addition and scalar multiplication.

Theorem

If the vector $\mathbf{v} \neq \mathbf{0}$*, then the vector* $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ *is a unit vector.*

Proof: (Case $\mathbf{v} \in \mathbb{R}^2$ only).

If $\mathbf{v} = \langle v_x, v_y \rangle \in \mathbb{R}^2$, then $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$, and

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{v_x}{|\mathbf{v}|}, \frac{v_y}{|\mathbf{v}|} \right\rangle.$$

This is a unit vector, since

$$|\mathbf{u}| = \left| \frac{\mathbf{v}}{|\mathbf{v}|} \right| = \sqrt{\left(\frac{v_x}{|\mathbf{v}|}\right)^2 + \left(\frac{v_y}{|\mathbf{v}|}\right)^2} = \frac{1}{|\mathbf{v}|}\sqrt{v_x^2 + v_y^2} = \frac{|\mathbf{v}|}{|\mathbf{v}|} = 1.$$

□

Addition and scalar multiplication.

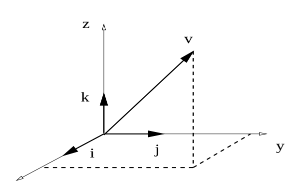
Theorem

Every vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *in* $\mathbb{R}^3$ *can be expressed in a unique way as a linear combination of vectors* $\mathbf{i} = \langle 1, 0, 0 \rangle$*,* $\mathbf{j} = \langle 0, 1, 0 \rangle$*,and* $\mathbf{k} = \langle 0, 0, 1 \rangle$ *as follows*

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.$$

Proof: Use the definitions of vector addition and scalar multiplication as follows,

$$\begin{aligned} \mathbf{v} &= \langle v_x, v_y, v_z \rangle \\ &= \langle v_x, 0, 0 \rangle + \langle 0, v_y, 0 \rangle + \langle 0, 0, v_z \rangle \\ &= v_x \langle 1, 0, 0 \rangle + v_y \langle 0, 1, 0 \rangle + v_z \langle 0, 0, 1 \rangle \\ &= v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}. \end{aligned}$$



□

Addition and scalar multiplication.

Example

Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$, that is,

$$\mathbf{v} = \langle (-1-1), (4-0), (5-3) \rangle = \langle -2, 4, 2 \rangle.$$

Then, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. $\lhd$

Example

Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

Solution: Since $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} = \sqrt{24}$, we conclude that $\mathbf{w} = -\frac{1}{\sqrt{24}} \langle -2, 4, 2 \rangle$. $\lhd$