Cartesian coordinates in space (Sect. 12.1).

- Overview of Multivariable Calculus.
- Cartesian coordinates in space.
- Right-handed, left-handed Cartesian coordinates.
- Distance formula between two points in space.
- Equation of a sphere.

Overview of Multivariable Calculus

Mth 132, Calculus I: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, differential calculus.

Mth 133, Calculus II: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, integral calculus.

Mth 234, Multivariable Calculus:

$$\left.\begin{array}{ll} f : \mathbb{R}^2 \to \mathbb{R}, & f(x,y) \\ f : \mathbb{R}^3 \to \mathbb{R}, & f(x,y,z) \end{array}\right\} \quad \text{scalar-valued.}$$

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \quad \} \quad \text{vector-valued.}$$

We study how to differentiate and integrate such functions.

The functions of Multivariable Calculus

Example

- An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature T of a plane surface, say a table. Each point (x, y) on the table is associated with a number, its temperature $T(x, y)$.
- An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature T of an object, say a room. Each point (x, y, z) in the room is associated with a number, its temperature $T(x, y, z)$.
- An example of a vector-valued function of one variable, $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, is the position function in time of a particle moving in space, say a fly in a room. Each time t is associated with the position vector $\mathbf{r}(t)$ of the fly in the room.

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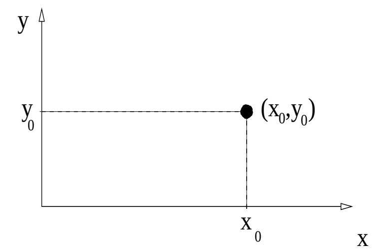
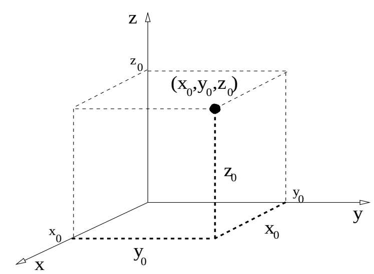
Cartesian coordinates.

Cartesian coordinates on $\mathbb{R}^2$: Every point on a plane is labeled by an ordered pair (x, y) by the rule given in the figure.



Cartesian coordinates in $\mathbb{R}^3$: Every point in space is labeled by an ordered triple (x, y, z) by the rule given in the figure.

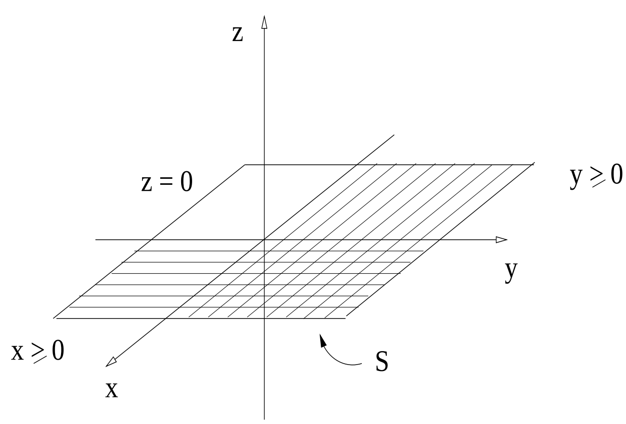


Cartesian coordinates.

Example

Sketch the set $S = \{x \geqslant 0,\ y \geqslant 0,\ z = 0\} \subset \mathbb{R}^3$.

Solution:



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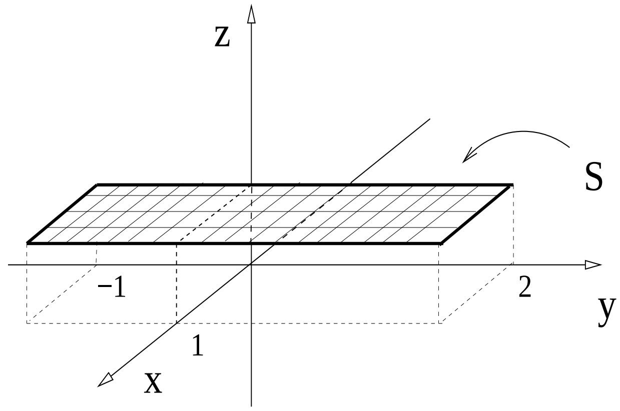
Cartesian coordinates.

Example

Sketch the set $S = \{0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ z = 1\} \subset \mathbb{R}^3$.

Solution:



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- Overview of vector calculus.
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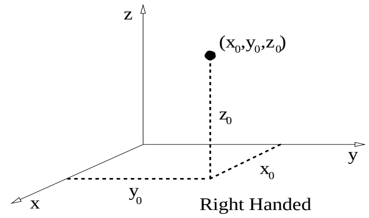
Right and left handed Cartesian coordinates.

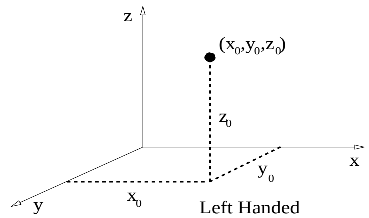
Definition

A Cartesian coordinate system is called *right-handed* (rh) iff it can be rotated into the coordinate system in the figure.



Definition

A Cartesian coordinate system is called *left-handed* (lh) iff it can be rotated into the coordinate system in the figure.

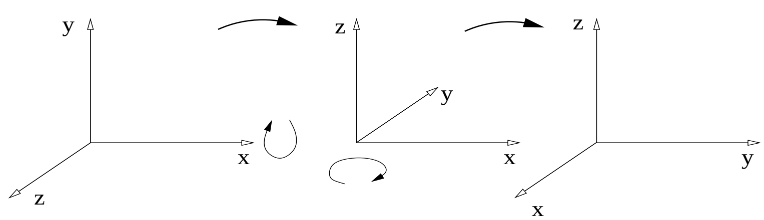


No rotation transforms a rh into a lh system.

Right and left handed Cartesian coordinates.

Example

This coordinate system is right-handed.



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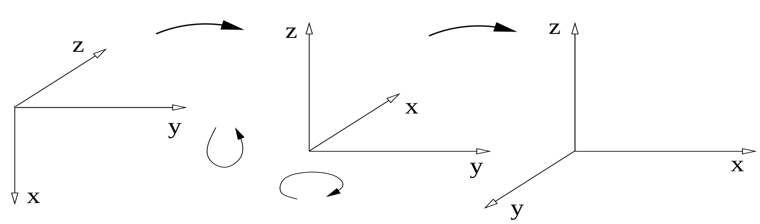
Example

This coordinate system is left handed.



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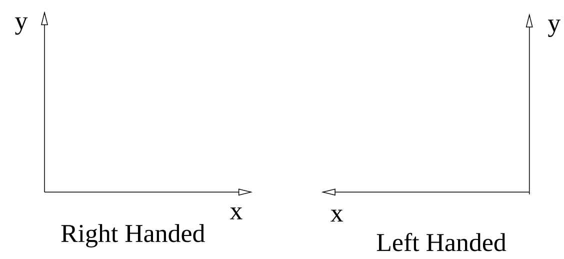
Right and left handed Cartesian coordinates

Remark: The same classification occurs in $\mathbb{R}^2$:



This classification is needed in $\mathbb{R}^3$ because:

- In $\mathbb{R}^3$ we will define the cross product of vectors, and this product has different results in rh or lh Cartesian coordinates.
- There is no cross product in $\mathbb{R}^2$.

In class we use rh Cartesian coordinates.

Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
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- **Distance formula between two points in space.**
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Distance formula between two points in space.

Theorem

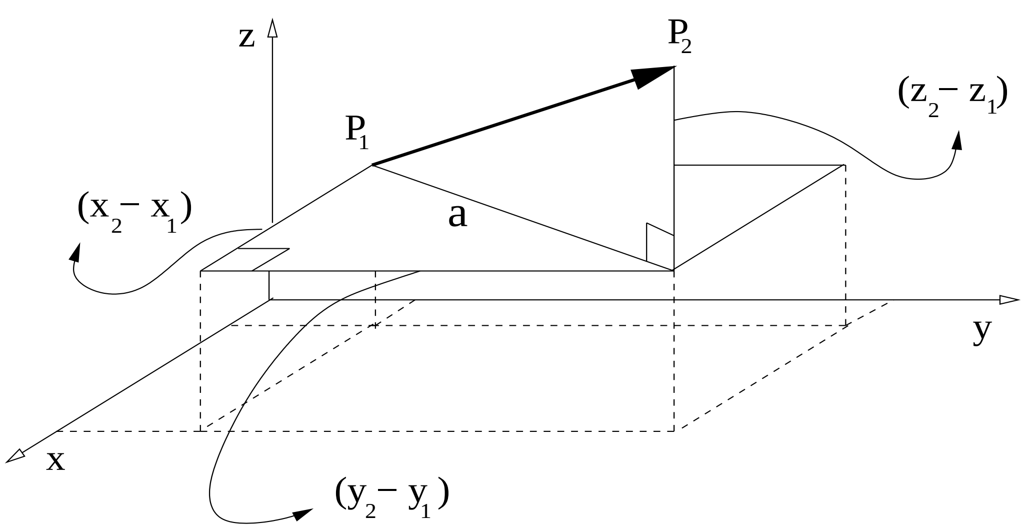
The distance $|P_1P_2|$ *between the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The distance between points in space is crucial to define the idea of limit to functions in space.

Proof.

Pythagoras Theorem.



$$|P_1P_2|^2 = a^2 + (z_2 - z_1)^2, \qquad a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

□

Distance formula between two points in space

Example

Find the distance between $P_1 = (1, 2, 3)$ and $P_2 = (3, 2, 1)$.

Solution:

$$|P_1P_2| = \sqrt{(3-1)^2 + (2-2)^2 + (1-3)^2}$$

$$|P_1P_2| = \sqrt{4+0+4} = \sqrt{8}$$

We conclude that

$$|P_1P_2| = 2\sqrt{2}.$$

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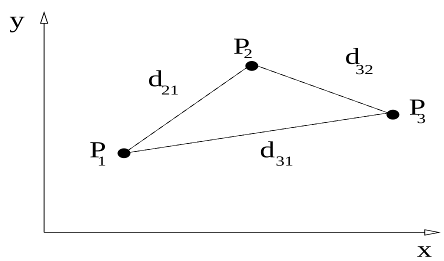
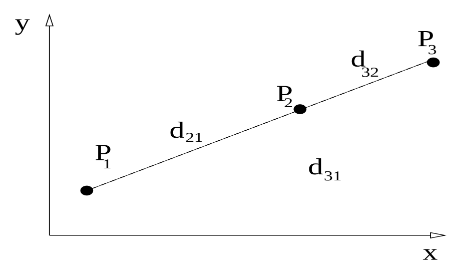
Distance formula between two points in space

Example

Use the distance formula to determine whether three points in space are collinear.

Solution:





$$d_{21} + d_{32} > d_{31}$$

Not collinear,

$$d_{21} + d_{32} = d_{31}$$

Collinear.

$\triangleleft$

Cartesian coordinates in space (12.1)

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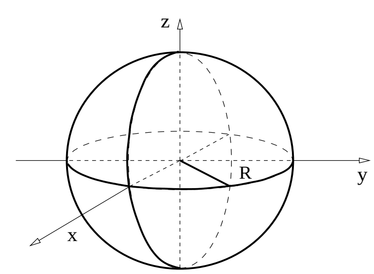
- **Equation of a sphere.**

A sphere is a set of points at fixed distance from a center.

Definition
A *sphere* centered at $P_0 = (x_0, y_0, z_0)$ of radius R is the set

$$S = \{P = (x, y, z) \ : \ |P_0 P| = R\}.$$



Remark: The point (x, y, z) belongs to the sphere S iff holds

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

("iff" means "if and only if.")

An open ball is a set of points contained in a sphere.

Definition

An *open ball* centered at $P_0 = (x_0, y_0, z_0)$ of radius R is the set

$$B = \{P = (x, y, z) \ : \ |P_0 P| < R\}.$$

A *closed ball* centered at $P_0 = (x_0, y_0, z_0)$ of radius R is the set

$$B = \{P = (x, y, z) \ : \ |P_0 P| \leqslant R\}.$$

Remark: The point (x, y, z) belongs to the open ball B iff holds

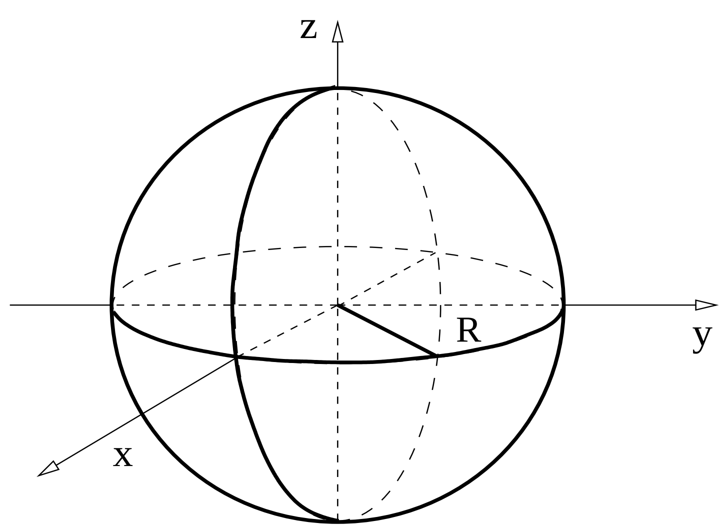
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < R^2.$$

Equation of a sphere

Example

Plot a sphere centered at $P_0 = (0, 0, 0)$ of radius $R > 0$.

Solution:



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Equation of a sphere

Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

Solution: Complete the square.

$$0 = x^2 + y^2 + 4y + z^2$$

$$0 = x^2 + \left[y^2 + 2\left(\frac{4}{2}\right)y + \left(\frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 + z^2$$

$$0 = x^2 + \left(y + \frac{4}{2}\right)^2 + z^2 - 4.$$

$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2.$$

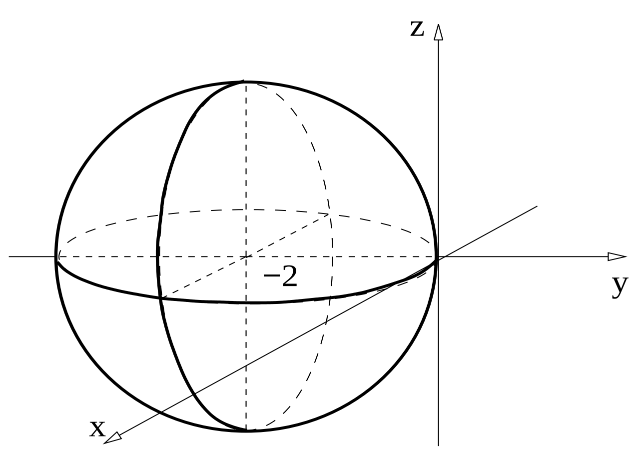
Equation of a sphere

Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

Solution: $x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2$.

Then, we conclude that $P_0 = (0, -2, 0)$ and $R = 2$. Therefore,



Equation of a sphere

Example

- *Given constants a, b, c, and $d \in \mathbb{R}$, show that*

$$x^2 + y^2 + z^2 - 2a\,x - 2b\,y - 2c\,z = d$$

is the equation of a sphere iff holds

$$d > -(a^2 + b^2 + c^2). \qquad (1)$$

- *Furthermore, show that if Eq. (1) is satisfied, then the expressions for the center P_0 and the radius R of the sphere are given by*

$$P_0 = (a, b, c), \qquad R = \sqrt{d + (a^2 + b^2 + c^2)}.$$

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