Plan: * Equations with singular points

* Euler differential eq.

\[(x-x_0)^2 y'' + \alpha (x-x_0) y' + \beta y = 0\]

* Solutions near \(x_0\)

* The roots of the indicial polynomial.

* (Equations with regular-singular points)

(5.4)
* Equations with Singular Points

Recall: \( x_0 \in \mathbb{R} \) is a singular point of

\[
P(x) y'' + Q(x) y' + R(x) y = 0 \quad (1)
\]

if \( P(x_0) = 0 \).

- We are interested in finding solutions to (1) near a singular point \( x_0 \).
- The order of the eq. changes near a singular point.

as \( x \to x_0 \):

- two l.d.i. solutions remain bounded
  - only one sol. remain bounded
  - none sol. remain bounded.

or
- It is known how to find solutions of eqns. near singular points in the case that the points are not so singular.

- Such points will be called regular singular points.

- Main example of an equation with regular singular points is the Euler differential eq.
The Euler equation

Def: Given constants \( x_0, \alpha, \beta \in \mathbb{R} \), the differential equation

\[
(x-x_0)^2 y'' + \alpha (x-x_0) y' + \beta y = 0
\]

is called the Euler equation.

Remarks:

- Eq. (2) has variable coefficients.
- Solutions of (2) are not of the form \( y = e^{rx} \)
- \( x_0 \in \mathbb{R} \) is a singular point of (2).
- We are interested in finding sols. to (2) arbitrary close to \( x_0 \).
- Particular case \( x_0 = 0 \) is

\[
x^2 y'' + \alpha x y' + \beta y = 0
\]

\( x_0 = 0 \) is a singular point.
* Main idea to find solutions to (3)

- Recall the constant coeff. case:

\[ y'' + a_1 y' + a_0 y = 0 \]  \hfill (4)

we looked for solutions, since the exponential can be canceled out of eq. (4)

\[ y = e^{r x} \]

so the number \( r \) of the characteristic eq.

\[ (r^2 + a_1 r + a_0) e^{r x} = 0 \]  \hfill (5)

- In the case of Euler eq.

\[ x^2 y'' + a x y' + b y = 0 \]

exponentials functions do not have the property in (4), since for \( y = e^{r x} \)

\[ (x^2 r^2 + a x r + b) e^{r x} = 0 \]

depends on \( x \).
Idea: Look for solutions $y(x) = x^r$

$y'(x) = r \cdot x^{r-1}$ \implies \boxed{x \cdot y' = r \cdot x^r}$

$y''(x) = r \cdot (r-1) \cdot x^{r-2}$ \implies \boxed{x^2 \cdot y'' = r \cdot (r-1) \cdot x^r}$

Introduce $y$ into $x^2 \cdot y'' + a \cdot x \cdot y' + \beta \cdot y = 0$

$\left( r \cdot (r-1) + a \cdot r + \beta \right) \cdot x^r = 0$

So $r$ must be solution of \boxed{$\Phi(r) = r \cdot (r-1) + a \cdot r + \beta = 0$}

Indicial eq. or Euler characteristic eq.

$r$ must be a root of $\Phi(r)$. 
Thm: Given constants $x_0, x, r \in \mathbb{R}$ consider the Euler eq.

$$\left(x-x_0\right)^2 y'' + \alpha \left(x-x_0\right) y' + \beta y = 0 \quad (6)$$

Let $\Gamma_1, \Gamma_2$ be roots of the indicial polynomial

$$q(r) = r(r-1) + \alpha r + \beta$$

(a) If $\Gamma_1 \neq \Gamma_2$, then the general sol. of (6) is

$$y(x) = c_1 \left|x-x_0\right|^\Gamma_1 + c_2 \left|x-x_0\right|^\Gamma_2$$

with $c_1, c_2$ constants.

(b) If $\Gamma_1 = \Gamma_2$, then the general sol. of (6) is

$$y(x) = c_1 \left|x-x_0\right|^\Gamma_1 + c_2 \ln\left|x-x_0\right| \left|x-x_0\right|^\Gamma_1$$

with $c_1, c_2$ constants.
Example: Find the general sol. of
\[ x^2 y'' + 4xy' + 2y = 0 \quad (7) \]
We look for sols. \( y = x^r \quad x > 0 \),
\[ x^2 r(r-1)x^{r-2} + 4x^r x^{r-1} + 2x^r = 0 \]
\[ (r(r-1) + 4r + 2)x^r = 0 \]
\( r \) must be root of the indicial polynomial
\[ g(r) = r(r-1) + 4r + 2 = 0 \]
\[ = r^2 + 3r + 2 = 0 \]
\[ r = \frac{-3 \pm \sqrt{9 - 8}}{2} = \frac{-3 \pm 1}{2} \]
\( r_1 = -1 \)
\( r_2 = -2 \)

\( y_1(x) = x^{-1} \)  
\( y_2(x) = x^{-2} \)  
Fundamental solutions of (7)  
for \( x > 0 \).
The Thom says that

\[
\begin{align*}
\varphi_1 &= 1 \times x^{-1} \\
\varphi_2 &= 1 \times x^{-2}
\end{align*}
\]

fundamental sols. for \(x \neq 0\).

The general sol. is

\[Y(x) = c_1 \times x^{-1} + c_2 \times x^{-2}\]

\(c_1, c_2 \in \mathbb{R}\)
Example: Find the general sol. of

\[ x^2 y'' - 3x y' + 4y = 0 \]

Sol:

we look for \( y = x^r \), \( x > 0 \), so

\[ (r(r-1) - 3r + 4) x^r = 0 \]

\( r \) must be root of the indicial polynomial

\[ g(r) = r(r-1) - 3r + 4 = 0 \]

\[ r^2 - 4r + 4 = 0 \]

\[ r = \frac{4 \pm \sqrt{16 - 16}}{2} \]

\[ r_1 = r_2 = 2 \]

(repeated roots)

\( y_1 = x^2 \)

\( y_2 = x^2 \ln(1x1) \), \( x \neq 0 \)

(fundamental solutions)

The general sol. is

\[ y(x) = c_1 x^2 + c_2 x^2 \ln(1x1) \]

\( x \neq 0 \).
Proof of Theorem (Existence) (case $x_0 > 0$)

Propose $y = x^r$, $x > 0$.

Then $r$ must be sol. of

$$g(r) = r(r-1) + 2r + \beta = 0$$

$$\Gamma_1 = \frac{-r - \sqrt{(r-1)^2 - 4\beta}}{2}$$

(1) $(x-1)^2 - 4\beta > 0 \implies$

This case includes:

$\Gamma_1, \Gamma_2 \in \mathbb{R}$

and

$$\begin{bmatrix}
\Gamma_1 = x^{\Gamma_1} \\
\Gamma_2 = x^{\Gamma_2}
\end{bmatrix} \in C$$
(2) \((a-1) \cdot 9 \beta = 0 \Rightarrow \Gamma_1 = \Gamma_2 = -\frac{(a-1)}{2}\)

\[ Y_1 = x^{\Gamma_1} \quad x > 0. \]

- Find a solution \(Y_2\) h.s. to \(Y_1\).
- Reduction of order method.

We look for

\[ Y_2 = \nu(x) \ x^{\Gamma_1} \]

\[ Y_2' = \nu' \ x^{\Gamma_1} + \nu \Gamma_1 \ x^{\Gamma_1-1} \]

\[ Y_2'' = \nu'' \ x^{\Gamma_1} + 2\Gamma_1 \nu' \ x^{\Gamma_1-1} \]

\[ + \nu \Gamma_1 (\Gamma_1 - 1) \ x^{\Gamma_1-2} \]

Introduce \(Y_2\) into

\[ x^2 \ Y_2'' + a \cdot x \ Y_2' + \beta \ Y_2 = 0 \]
\[ 0 = V'' \times x^{r+2} + 2 \Gamma_1 V' \times x^{\Gamma+1} + V \Gamma_1 (\Gamma_1 - 1) x^\Gamma \\
+ \alpha (V' x^{\Gamma+1} + V \Gamma_1 x^\Gamma) \\
+ \beta V x^\Gamma \]

\[ 0 = V'' \times x^{r+2} + (2 \Gamma_1 + \alpha) V' \times x^{\Gamma+1} \\
+ (\Gamma_1 (\Gamma_1 - 1) + \alpha \Gamma_1 + \beta) V x^\Gamma \]

**Recall:**

\[ \Gamma_1 (\Gamma_1 - 1) + \alpha \Gamma_1 + \beta = 0 \]

\[ \Gamma_1 : \text{root of } \Phi \]

\[ \Gamma_1 = -\frac{(\Gamma_1 - 1)}{2} \implies 2 \Gamma_1 + \alpha = 1 \]

\[ \Gamma_1 \text{ the only root of } \Phi \]

\[ 0 = V'' \times x^{r+2} + V' \times x^{\Gamma+1} \]

\[ V'' x + V' = 0 \]
\[ u = v' \implies xu' + u = 0 \]

\[ xu' = -u \]

\[ \frac{u'}{u} = -\frac{1}{x} \]

\[ \ln u = -\ln x = \ln \left( \frac{1}{x} \right) \]

\[ u = \frac{1}{x} \]

\[ u = v' \]

\[ v' = \frac{1}{x} \implies v(x) = \ln(x) \quad x > 0 \]

\[ y_1 = x^\gamma, \quad y_2 = x^\gamma \ln(x) \quad x > 0. \]
For $x < 0$, introduce the change of variable: $x = -X \Rightarrow x > 0$.

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \rho y = 0
\]

$x = -X$, \quad \frac{d}{dx} = \frac{d}{dX} \frac{dX}{dx} = - \frac{d}{dX}

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{d y}{dX} \right)
\]

\[
= \frac{d}{dX} \left( - \frac{d y}{dX} \right)
\]

\[
= (-1)^2 \frac{d^2 y}{dX^2}
\]

\[
\frac{d y}{dX} = - \frac{d y}{dX}
\]

\[
\frac{d^2 y}{dX^2} = \frac{d^2 y}{dx^2}
\]

\[
x^2 \frac{d^2 y}{dx^2} + x \ x \frac{dy}{dx} + \rho y = 0 \quad x > 0
\]

\[
\gamma_1 = x^\frac{5}{2}, \quad \gamma_2 = x^\frac{3}{2}, \quad x = -X
\]
**Summary:**
\[
\Gamma_1 \neq \Gamma_2
\]

\[x > 0 : \quad y_1 = x^{\Gamma_1}, \quad y_2 = x^{\Gamma_2}\]

\[-x : \quad y_1 = (-x)^{\Gamma_1}, \quad y_2 = (-x)^{\Gamma_2}\]

\[
\Gamma_1 = \Gamma_2
\]

\[x > 0 : \quad y_1 = x^{\Gamma_1}, \quad y_2 = x^{\Gamma_2} \ln(x)\]

\[-x : \quad y_1 = (-x)^{\Gamma_1}, \quad y_2 = (-x)^{\Gamma_2} \ln(-x)\]

**Therefore:**
\[
y_1 = 1x^\Gamma_1, \quad y_2 = 1x^\Gamma_2 \quad , \quad \Gamma_1 \neq \Gamma_2 \quad , \quad x \neq 0.
\]

\[
y_1 = 1x^\Gamma_1, \quad y_2 = 1x^\Gamma_2 \ln(|x|) \quad , \quad \Gamma_1 = \Gamma_2 \quad , \quad x \neq 0.
\]
Remark: The case of complex roots

The Euler eq.

\[ x^2 y'' + ax y' + by = 0 \quad (6) \]

In the case that

\[ q(r) = r(r-1) + ar + b \]

has roots

\[ r = -\frac{(a-1)}{2} \pm \frac{1}{2} \sqrt{(a-1)^2 - 4b} \]

with \( (a-1)^2 - 4b \leq 0 \)

Denote

\[ \mu = \sqrt{4b - (a-1)^2} \]

\[ \lambda = -\frac{(a-1)}{2} \]

The roots have the form

\[ r_1 = \lambda + i\mu \]

\[ r_2 = \lambda - i\mu \]
Proposition:

Real valued fundamental sols. of (b) in the case that \( \xi \) has complex roots are:

\[
\begin{align*}
\bar{\chi}_1 &= 1|x|^2 \cos \left[ \nu \ln(1|x|) \right] \\
\bar{\chi}_2 &= 1|x|^2 \sin \left[ \nu \ln(1|x|) \right]
\end{align*}
\]
Proof: Given \( \bar{y}_1, \bar{y}_2 \), introduce:

\[
\begin{align*}
\bar{y}_1 & = \frac{1}{2} (\bar{y}_1 + \bar{y}_2) \\
\bar{y}_2 & = \frac{1}{2} (\bar{y}_1 - \bar{y}_2)
\end{align*}
\]

\[
\begin{align*}
\bar{y}_1 & = 1 \times l^3 + 2 \times l^4 \\
& = 1 \times l^3 + 1 \times l^4 = 1 \times l^3 + \ln(1 \times l^4) \\
& = 1 \times l^3 + 2 \mu \ln l^4 \\
& = 1 \times l^3 e^{2 \mu \ln l^4} \quad \text{(Euler eq.)}
\end{align*}
\]

\[
\begin{align*}
\bar{y}_1 & = 1 \times l^3 \left[ \cos (\mu \ln l^4) + i \sin (\mu \ln l^4) \right] \\
\bar{y}_2 & = 1 \times l^3 \left[ \cos (\mu \ln l^4) - i \sin (\mu \ln l^4) \right] \\
\end{align*}
\]

\[
\begin{align*}
\bar{y}_1 & = 1 \times l^3 \cos (\mu \ln l^4) \\
\bar{y}_2 & = 1 \times l^3 \sin (\mu \ln l^4)
\end{align*}
\]
Example: Find real-valued fundamental sols. of
\[ x^2 y'' - 3x y' + 13 y = 0 \]

Sols:

\[ \gamma^2 - 4\gamma + 13 = 0 \]

\[ \gamma = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} \]

\[ \gamma_1 = 2 + 3i \quad \gamma_2 = 2 - 3i \]

Complex-val. sols.

\[ y_1 = x^2 \cos(3 \ln|x|) \quad y_2 = x^2 \sin(3 \ln|x|) \]

Real-valued fundamental sols.