Plan: 
* $y'' + a_y' + a_0y = 6(t)$.
* Method of undetermined coefficients
* Examples.

Notation: Given functions $P, Q$,

$$L(y) = y'' + P(t)y' + Q(t)y.$$ 

So, the equation

$$y'' + P(t)y' + Q(t)y = f(t)$$

can be written as

$$L(y) = f$$ non-homogeneous

$$L(y) = 0$$ homogeneous.
Preliminary results on \( L(y) = y'' + p(t) y' + q(t) y \)

**Proposition:** For all continuously differentiable functions \( y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R} \) and all \( c_1, c_2 \in \mathbb{R} \) holds

\[
L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)
\]

**Proof:**

\[
L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)'' + p(t) (c_1 y_1 + c_2 y_2)'
\]
\[
+ q(t) (c_1 y_1 + c_2 y_2)
\]
\[
= c_1 y_1'' + c_1 p(t) y_1' + c_1 q(t) y_1
\]
\[
+ c_2 y_2'' + c_2 p(t) y_2' + c_2 q(t) y_2
\]
\[
= c_1 L(y_1) + c_2 L(y_2)
\]
Propos. Let \( L(y) = y'' + p(t) y' + q(t) y \).

If \( y_1, y_2 \) are fundamental solns. of
\[
L(y) = 0, \tag{1}
\]
and \( y_p \) is a solution of
\[
L(y_p) = f, \tag{2}
\]
then any other solution of (2) is given by
\[
y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \tag{3}
\]
where \( c_1, c_2 \in \mathbb{R} \).

Notation: The expression for \( y \) in (3) is called the general solution of the non-homogeneous of (2).
Propos. If \( f(t) = f_1(t) + \cdots + f_n(t) \), \( n \geq 1 \), and there exist

\[ \gamma_1(t), \ldots, \gamma_n(t) \]

such that

\[ L(\gamma_i) = f_i, \quad i = 1, \ldots, n, \]

then

\[ \gamma_p(t) = \gamma_1(t) + \cdots + \gamma_n(t) \]

is solution of

\[ L(\gamma_p) = f. \]
The method of undetermined coefficients

This is a method to find solutions to linear, non-homogeneous equations.

It consists in guessing the solution $y_p$ of the non-homogeneous equation $L(y_p) = f$ for particular source functions $f$. 
* Summary of the method.

- Problem: Given \[ L(y) = y'' + a_1 y' + a_0 y, \]
  with \( a_1, a_2 \in \mathbb{R}, \) find all solutions to \[ L(y) = f. \] (4)

1. Find the general solution of the homogeneous eq.

\[ L(y_h) = 0. \]

2. If \( f(t) = f_1(t) + \ldots + f_n(t), \quad n \geq 1, \)
then look for \( y_{p_i}, \) solution of \[ L(y_{p_i}) = f_i. \]

Once \( y_{p_i} \) are found, a particular solution of (4) is

\[ y_p = y_{p_1} + \ldots + y_{p_n}. \]
(3) Given \( f_i(t) \), guess \( y_i(t) \) as follows:

<table>
<thead>
<tr>
<th>( f_i(t) ) (given)</th>
<th>( y_i(t) ) (guess) (( k ) not given)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ke^{at} )</td>
<td>( Ke^{at} )</td>
</tr>
<tr>
<td>( Kt^m, m \geq 0 )</td>
<td>( km t^m + km_{-1} t^{m-1} + \ldots + k_0 )</td>
</tr>
<tr>
<td>( K \cos(at) )</td>
<td>( K \cos(at) + k_2 \sin(at) )</td>
</tr>
<tr>
<td>( K \sin(at) )</td>
<td>( K \cos(at) + k_2 \sin(at) )</td>
</tr>
<tr>
<td>( Ke^{at} \cos(bt) )</td>
<td>( e^{at} (K \cos(bt) + k_2 \sin(bt)) )</td>
</tr>
<tr>
<td>( Ke^{at} \sin(bt) )</td>
<td>( e^{at} (K \cos(bt) + k_2 \sin(bt)) )</td>
</tr>
<tr>
<td>( Kt^m e^{at} )</td>
<td>( e^{at} (km t^m + \ldots + k_0) )</td>
</tr>
</tbody>
</table>
(4) If any guess \( y_{p_i} \) above satisfies the homogeneous eq.

\[ L(y_{p_i}) = 0, \]

then change the guess to

\[ t^s y_{p_i}, \quad s \geq 1, \]

such that

\[ L(t^s y_{p_i}) \neq 0. \]

(5) Impose the condition

\[ L(y_{p_i}) = f_i \]

to find the constants \( k_1, \ldots, k_m \).

(6) The general solution of \( L(y) = f \) is

\[ y(t) = y_h(t) + y_{p_1}(t) + \ldots + y_{p_n}(t). \]
Example: Find all solutions to

\[ y'' - 3y' - 4y = 3e^{2t} \]

Solution:

\[ L(y) = y'' - 3y' - 4y \quad \text{and} \quad f(t) = 3e^{2t} \]

First, find sols. to \[ L(y_h) = 0. \]

\[ p(t) = r^2 - 3r - 4 = 0 \]

\[ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 + 16}}{2} = \frac{3 \pm 5}{2} \]

\[ r_1 = 4 \quad \text{and} \quad r_2 = -1 \]

\[ y_h(t) = c_1 e^{4t} + c_2 e^{-t} \]

- Table says: Guess \[ y_p(t) = k e^{2t} \]

- Since \[ L(y_p) \neq 0 \], we do not modify the guess.
- Introduce $y_p$ in: $L(y_p) = 0$

\[(k^4 - 3k^2 - 4k)e^{2t} = 3e^{2t}\]

\[4k - 6k - 4k = 3\]

\[-6k = 3\]

\[k = -\frac{1}{2}\]

\[y_p(t) = -\frac{1}{2}e^{2t}\]

- The general solution of the non-homogeneous eq. is

\[y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}\]
Example: Find the general sol. of
\[ y'' - 3y' - 4y = 3e^{4t} \]

Sol:
We know that the general sol. to the homogeneous eq. is
\[ y_h = c_1 e^{4t} + c_2 e^{-t} \]

Following the table, the guess \( y_p \) is
\[ y_p = k e^{4t} \]

However, this guess satisfies:
\[ L (y_p) = 0 \]

so we modify the guess to:
\[ y_p (t) = k t e^{4t} \]
\[ y_p(t) = k e^{4t} + 4kt e^{4t} \]
\[ y''(t) = 4ke^{4t} + 4ke^{4t} + 16kt e^{4t} \]

Introduce \( y_p \) into \( L(y_p) = 0 \).

\[
\left[(8k + 16kt) - 3(k+4kt) - 4kt\right]e^{4t} = 3e^{4t}
\]

\[
\left[(8+16t) - (3+12t) - 4t\right]k = 3
\]

\[
\left[5 + (16-12-4)t\right]k = 3
\]

So:
\[
y_p(t) = \frac{3}{5}te^{4t}\]

and
\[
y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{5}te^{4t}\]
Example \[ \text{Find the general sol. of} \]
\[
y'' - 3y' - 4y = 2 \sin(t)
\]

\[ \text{Sol.:} \]

- The sols. to the homogeneous eq. are

\[
y_h(t) = c_1 e^{2t} + c_2 e^{-t}
\]

- The table says that a guess for \( y_p \) is

\[
y_p(t) = k_1 \cos(t) + k_2 \sin(t).
\]

- Check: \( L(y_p) \neq 0 \).

- Compute:

\[
y_p'(t) = -k_1 \sin(t) + k_2 \cos(t)
\]

\[
y_p''(t) = -k_1 \cos(t) - k_2 \sin(t).
\]
- Introduce $\gamma$ into $L(\gamma) = L$.

\[- r_1 \cos(t) - r_2 \sin(t) \]
\[- 3 \left( - r_1 \sin(t) + r_2 \cos(t) \right) \]
\[- 4 \left( r_1 \cos(t) + r_2 \sin(t) \right) = 2 \sin(t) \]

\[- r_1 - 3 r_2 - 4 r_1 \cos(t) \]
\[+ (-r_2 + 3 r_1 - 4 r_2) \sin(t) = 2 \sin(t) \]

\[- 5 r_1 - 3 r_2 \cos(t) + (3 r_1 - 5 r_2) \sin(t) = 2 \sin(t) \]

\[
\begin{array}{c|c}
 t=0 & -5 r_1 - 3 r_2 = 0 \\
t=\frac{\pi}{2} & 3 r_1 - 5 r_2 = 2 \\
\end{array}
\]
1. \( r_1 = -\frac{3}{5} \) \( r_2 \).

2. \[
\left( \begin{array}{cc}
3 & -\frac{3}{5} \\
-\frac{9}{5} & -5
\end{array} \right) k_2 = 2
\]

3. \[
\left( \begin{array}{cc}
-\frac{9}{5} & -5
\end{array} \right) k_2 = 2
\]

4. \[ -\frac{(9 + 23)}{5} k_2 = 2 \]

5. \[
k_2 = -\frac{10}{34} \implies k_2 = -\frac{5}{17}
\]

6. \[
k_1 = \frac{3}{17}
\]

So:

7. \[
Y_p(t) = \frac{1}{17} \left[ 3 \cos(t) - 5 \sin(t) \right]
\]

And:

8. \[
Y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17} \left[ 3 \cos(t) - 5 \sin(t) \right]
\]
Example: Find the general sol. of
\[ y'' - 3y' - 4y = 3e^{2t} + 25\sin(t) \]

Sol.:
We know that the sol. \( y \) is given by
\[ y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t) \]
where,
\[ y_h(t) = c_1 e^{4t} + c_2 e^{2t} \]
and \( y_{p_1} \) is sol. of
\[ L(y_{p_1}) = 3e^{2t} \]
and \( y_{p_2} \) is sol. of
\[ L(y_{p_2}) = 25\sin(t) \]

We have found that,
\[ y_{p_1}(t) = -\frac{1}{2} e^{2t} \]
\[ y_{p_2}(t) = \frac{1}{17} (3\cos(t) - 5\sin(t)) \]
So: \( y(t) = e^{4t} + e^{-t} - \frac{1}{2} e^{2t} \)

\[
+ \frac{1}{17} \left( 3 \cos(t) - 5 \sin(t) \right)
\]

**Examples:**

For: \( y'' - 3y' - 4y = 3e^{2t} \sin(t) \)

**guess:** \( y_p(t) = \left[ k_1 \sin(t) + k_2 \cos(t) \right] e^{2t} \)

For: \( y'' - 3y' - 4y = 3t^2 e^{2t} \)

**guess:** \( y_p(t) = (k_0 + k_1 t + k_2 t^2) e^{2t} \)

For: \( y'' - 3y' - 4y = 3t \sin(t) \)

**guess:** \( y_p(t) = (1 + k_1 t)(k_2 \sin(t) + k_3 \cos(t)) \)