1. (15 points) Find the integrating factor that converts the equation below for the unknown $y$ into an exact equation, where

$$y' + ty + y^2 + \frac{y}{t} = 0.$$ 

You do not need to find the solution, only the integrating factor.

\[
\begin{align*}
(1 + t^2) y' + \left( y^2 + \frac{y}{t} \right) &= 0 \\
N &= 1 + t^2 y \
M &= y^2 + \frac{y}{t} \\
\frac{1}{N} \left( \delta_y M - \delta_t N \right) &= \frac{1}{1 + t^2 y} \left( 2y + \frac{t}{t} - y \right) \\
&= \frac{1}{1 + ty} \left( \frac{1}{t} + y \right) \\
&= \frac{1}{t} \left( \frac{1}{1 + ty} \right) \\
&= \frac{1}{t} \\
\mu &= t \\
\mu \ln \mu &= \ln t \\
\mu &= t
\end{align*}
\]
2. (17 points) Find all solutions $y$ to the initial value problem

$$y' = -\frac{3}{t} y + \frac{\cos(\pi t)}{t^2}, \quad y(1) = \frac{-1}{\pi^2}, \quad t > 0.$$
3. (17 points) A tank initially contains 200 liters of water with 50 lb of salt. The tank is rinsed with fresh water flowing in at a rate of 2 liters per minute and leaving the tank at the same rate. The water in the tank is well-stirred. Find the time such that the amount of salt in the tank is 10% the initial amount.

\[
\begin{align*}
\Gamma = \Gamma_0 = \Gamma = 2 \text{ \frac{L}{min}} \\
q_1 = 0 \quad V_0 = 200 \text{ L} \\
Q_0 = 50 \text{ lb}
\end{align*}
\]

\[
V'(t) = 0 \Rightarrow V(t) = V_0
\]

\[
Q'(t) = \Gamma \frac{Q}{V_0} \Rightarrow Q(t) = Q_0 e^{-\frac{\Gamma}{V_0} t}
\]

\[
5 \times Q_0 = \frac{Q_0}{10} \Rightarrow Q(t_1) = Q_0 e^{-\frac{\Gamma}{V_0} t_1}
\]

\[
\ln \frac{1}{10} = -\frac{\Gamma}{V_0} t_1
\]

\[
\ln 10 = \frac{\Gamma}{V_0} t_1 \Rightarrow t_1 = \frac{V_0}{\Gamma} \ln 10
\]

\[
t_1 = 100 \ln 10
\]
4. (17 points) Find all solutions $y$ to the initial value problem

$$(y + t^2 y) y' = 2t, \quad y(0) = -2.$$ 

\[
(1 + t^2) y y' = 2t \quad \text{Separable eq.}
\]

\[
y y' = \frac{2t}{1 + t^2}
\]

\[
\int y y' \, dt = \int \frac{2t}{1 + t^2} \, dt + c
\]

$u = y(t) \quad v = 1 + t^2$

\[
du = y' \, dt \quad dv = 2t \, dt
\]

\[
\int u \, du = \int \frac{dv}{v} + c
\]

\[
\frac{u^2}{2} = \ln(v) + c
\]

\[
\left. \frac{y^2}{2} \right|_0 = \ln (1 + t^2) + c
\]

\[
\left. \frac{(-2)^2}{2} \right|_0 = \ln (1) + c
\]

\[c = 2\]
5. (17 points) Find an explicit expression for all solutions $y$ to the initial value problem

$$y'(t) = \frac{2y - t^2}{y - 2t}$$

$$y(1) = 0.$$

**Exact 2**

$$(y - 2t) y' + (t^2 - 2y) = 0$$

$N = y - 2t \quad \Rightarrow \quad \tau N = -2$

$M = t^2 - 2y \quad \Rightarrow \quad \tau M = -2$

$\frac{\partial y}{\partial y} = N \quad \Rightarrow \quad \tau \frac{\partial y}{\partial y} = y - 2t \quad \Rightarrow \quad y = \frac{y^2}{2} - 2ty + \frac{t^2}{3}$

$\tau y = M$

$-2y + \tau y' = \tau \frac{\partial y}{\partial y} = M = t^2 - 2y$

$y' = t^2 \quad \Rightarrow \quad y(t) = \frac{t^3}{3}$

$$\frac{y^2}{2} - 2ty + \frac{t^3}{3} = c$$

$y(1) = 0$

$0 = 0 + \frac{1}{3} = c$

$$\frac{y^2}{2} - 2ty + \frac{t^3}{3} = \frac{1}{3}$$
6. (17 points) Find an explicit expression for all solutions $y$ to the differential equation

$$t^2 y' = ty - y^3.$$