Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
- Partial derivatives and directional derivatives.
- Directional derivative of functions of three variables.
- The gradient vector and directional derivatives.
- Properties of the gradient vector.

Directional derivative of functions of two variables.

Remark: The directional derivative generalizes the partial derivatives to any direction.

Definition
The direction derivative of the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y \rangle$ is given by

$$ (D_\mathbf{u}f)_{P_0} = \lim_{t \to 0} \frac{1}{t} [f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)] , $$

if the limit exists.

Notation: The directional derivative is also denoted as

$$ \left( \frac{df}{dt} \right)_{\mathbf{u}, P_0} . $$
Directional derivatives generalize partial derivatives.

Example
The partial derivatives $f_x$ and $f_y$ are particular cases of directional derivatives $(D_u f)_{P_0} = \lim_{t \to 0} \frac{1}{t}[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)]:$

- $u = \langle 1, 0 \rangle = i$, then $(D_i f)_{P_0} = f_x(x_0, y_0)$.
- $u = \langle 0, 1 \rangle = j$, then $(D_j f)_{P_0} = f_y(x_0, y_0)$.

Remark: The condition $|u| = 1$ implies that the parameter $t$ in the line $r(t) = \langle x_0, y_0 \rangle + u t$ is the distance between the points $(x(t), y(t)) = (x_0 + u_x t, y_0 + u_y t)$ and $(x_0, y_0)$.

Proof.

\[ d = |\langle x - x_0, y - y_0 \rangle| = |\langle u_x t, u_y t \rangle| = |t| |u|, \]

that is, $d = |t|$. \hfill \Box

Remark: The directional derivative of $f(x, y)$ at $P_0 = (x_0, y_0)$ along $u$, denoted as $(D_u f)_{P_0}$, is the pointwise rate of change of $f$ with respect to the distance along the line parallel to $u$ passing through $(x_0, y_0)$. 
Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
- **Partial derivatives and directional derivatives.**
- Directional derivative of functions of three variables.
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- Properties of the gradient vector.

Remark: The directional derivative \((D_u f)_{P_0}\) is the derivative of \(f\) along the line \(r(t) = (x_0, y_0) + u t\).

Theorem

*If the function \(f : D \subset \mathbb{R}^2 \to \mathbb{R}\) is differentiable at \(P_0 = (x_0, y_0)\) and \(u = \langle u_x, u_y \rangle\) is a unit vector, then*

\[
(D_u f)_{P_0} = f_x(x_0, y_0) u_x + f_y(x_0, y_0) u_y.
\]
Directional derivative and partial derivatives.

Proof. The line \( r(t) = (x_0, y_0) + (u_x, u_y) \ t \) has parametric equations:
\[ x(t) = x_0 + u_x t \] and \( y(t) = y_0 + u_y t; \)
Denote \( f \) evaluated along the line as \( \hat{f}(t) = f(x(t), y(t)). \)
Now, on the one hand, \( \hat{f}'(0) = (D_u f)_{P_0}, \) since
\[
\hat{f}'(0) = \lim_{t \to 0} \frac{1}{t} \left[ \hat{f}(t) - \hat{f}(0) \right],
\]
\[ = \lim_{t \to 0} \frac{1}{t} \left[ f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0) \right], \]
\[ = D_u f(x_0, y_0). \]
On the other hand, the chain rule implies:
\[
\hat{f}'(0) = f_x(x_0, y_0) x'(0) + f_y(x_0, y_0) y'(0).
\]
Therefore, \( (D_u f)_{P_0} = f_x(x_0, y_0) u_x + f_y(x_0, y_0) u_y. \) \( \square \)

Directional derivative and partial derivatives.

Example
Compute the directional derivative of \( f(x, y) = \sin(x + 3y) \) at the point \( P_0 = (4, 3) \) in the direction of vector \( v = (1, 2). \)

Solution: We need to find a unit vector in the direction of \( v. \)
Such vector is \( u = \frac{v}{|v|} \Rightarrow u = \frac{1}{\sqrt{5}} (1, 2). \)
We now use the formula \( (D_u f)_{P_0} = f_x(x_0, y_0) u_x + f_y(x_0, y_0) u_y. \)
That is, \( (D_u f)_{P_0} = \cos(x_0 + 3y_0)(1/\sqrt{5}) + 3 \cos(x_0 + 3y_0)(2/\sqrt{5}). \)
Equivalently, \( (D_u f)_{P_0} = (7/\sqrt{5}) \cos(x_0 + 3y_0). \)
Then, \( (D_u f)_{P_0} = (7/\sqrt{5}) \cos(10). \) \( \triangle \)
Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
- Partial derivatives and directional derivatives.
- **Directional derivative of functions of three variables.**
- The gradient vector and directional derivatives.
- Properties of the the gradient vector.

**Definition**

The *directional derivative* of the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0, z_0) \in D$ in the direction of a unit vector $u = \langle u_x, u_y, u_z \rangle$ is given by

\[
(D_u f)_{P_0} = \lim_{t \to 0} \frac{1}{t} \left[ f(x_0 + u_x t, y_0 + u_y t, z_0 + u_z t) - f(x_0, y_0, z_0) \right],
\]

if the limit exists.

**Theorem**

*If the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0, z_0)$ and $u = \langle u_x, u_y, u_z \rangle$ is a unit vector, then*

\[
(D_u f)_{P_0} = f_x(x_0, y_0, z_0) u_x + f_y(x_0, y_0, z_0) u_y + f_z(x_0, y_0, z_0) u_z.
\]
Example
Find \((D_u f)_{P_0}\) for \(f(x, y, z) = x^2 + 2y^2 + 3z^2\) at the point \(P_0 = (3, 2, 1)\) along the direction given by \(v = \langle 2, 1, 1 \rangle\).

Solution: We first find a unit vector along \(v\),

\[ u = \frac{v}{|v|} \Rightarrow u = \frac{1}{\sqrt{6}} \langle 2, 1, 1 \rangle. \]

Then, \((D_u f)\) is given by \((D_u f) = (2x)u_x + (4y)u_y + (6z)u_z\).

We conclude, \((D_u f)_{P_0} = (6) \frac{2}{\sqrt{6}} + (8) \frac{1}{\sqrt{6}} + (6) \frac{1}{\sqrt{6}},\)

that is, \((D_u f)_{P_0} = \frac{26}{\sqrt{6}}.\) \(\triangleright\)
The gradient vector and directional derivatives.

Remark: The directional derivative of a function can be written in terms of a dot product.

- In the case of 2 variable functions: $D_uf = f_x u_x + f_y u_y$
  $$D_uf = (\nabla f) \cdot u, \quad \text{with} \quad \nabla f = \langle f_x, f_y \rangle.$$

- In the case of 3 variable functions: $D_uf = f_x u_x + f_y u_y + f_z u_z$,
  $$D_uf = (\nabla f) \cdot u, \quad \text{with} \quad \nabla f = \langle f_x, f_y, f_z \rangle.$$

The gradient vector and directional derivatives.

Definition

The gradient vector of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at any point $(x, y) \in D$ is the vector $\nabla f = \langle f_x, f_y \rangle$.

The gradient vector of a differentiable function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at any point $(x, y, z) \in D$ is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$.

Notation:

- For two variable functions: $\nabla f = f_x \mathbf{i} + f_y \mathbf{j}$.
- For two variable functions: $\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$.

Theorem

If $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3$, is a differentiable function and $\mathbf{u}$ is a unit vector, then,

$$D_uf = (\nabla f) \cdot \mathbf{u}.$$
The gradient vector and directional derivatives.

Example
Find the gradient vector at any point in the domain of the function $f(x, y) = x^2 + y^2$.

Solution: The gradient is $\nabla f = \langle f_x, f_y \rangle$, that is, $\nabla f = \langle 2x, 2y \rangle$. ◁

Remark:
$\nabla f = 2r$,
with $r = \langle x, y \rangle$.

Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
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Properties of the gradient vector.

**Remark:** If θ is the angle between \( \nabla f \) and \( \mathbf{u} \), then holds

\[
D_u f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos(\theta).
\]

The formula above implies:

- The function \( f \) increases the most rapidly when \( \mathbf{u} \) is in the direction of \( \nabla f \), that is, \( \theta = 0 \). The maximum increase rate of \( f \) is \( |\nabla f| \).
- The function \( f \) decreases the most rapidly when \( \mathbf{u} \) is in the direction of \( -\nabla f \), that is, \( \theta = \pi \). The maximum decrease rate of \( f \) is \( -|\nabla f| \).
- The function \( f \) does not change along level curve or surfaces, that is, \( D_u f = 0 \). Therefore, \( \nabla f \) is perpendicular to the level curves or level surfaces.

Properties of the gradient vector.

**Example**

Find the direction of maximum increase of the function \( f(x, y) = \frac{x^2}{4} + \frac{y^2}{9} \) at an arbitrary point \((x, y)\), and also at the points \((1, 0)\) and \((0, 1)\).

**Solution:** The direction of maximum increase of \( f \) is the direction of its gradient vector:

\[
\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle.
\]

At the points \((1, 0)\) and \((0, 1)\) we obtain, respectively,

\[
\nabla f = \left\langle \frac{1}{2}, 0 \right\rangle, \quad \nabla f = \left\langle 0, \frac{2}{9} \right\rangle.
\]
Example

Given the function \( f(x, y) = x^2/4 + y^2/9 \), find the equation of a line tangent to a level curve \( f(x, y) = 1 \) at the point \( P_0 = (1, -3\sqrt{3}/2) \).

Solution: We first verify that \( P_0 \) belongs to the level curve \( f(x, y) = 1 \). This is the case, since
\[
\frac{1}{4} + \frac{(9)(3)}{4} \frac{1}{9} = 1.
\]
The equation of the line we look for is
\[
\mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t \left\langle v_x, v_y \right\rangle,
\]
where \( \mathbf{v} = \left\langle v_x, v_y \right\rangle \) is tangent to the level curve \( f(x, y) = 1 \) at \( P_0 \).

Therefore, \( \mathbf{v} \perp \nabla f \) at \( P_0 \). Since,
\[
\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle \quad \Rightarrow \quad (\nabla f)_{P_0} = \left\langle \frac{1}{2}, -\frac{2 \cdot 3\sqrt{3}}{9} \right\rangle = \left\langle \frac{1}{2}, -\frac{1}{\sqrt{3}} \right\rangle.
\]

Therefore,
\[
0 = \mathbf{v} \cdot (\nabla f)_{P_0} \quad \Rightarrow \quad \frac{1}{2} v_x = \frac{1}{\sqrt{3}} v_y \quad \Rightarrow \quad \mathbf{v} = \left\langle 2, \sqrt{3} \right\rangle.
\]
The line is \( \mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t \left\langle 2, \sqrt{3} \right\rangle. \triangleq \)
Properties of the gradient vector.

\[ \nabla f = \langle \frac{1}{2}, 0 \rangle, \quad \nabla f = \langle 0, \frac{2}{9} \rangle, \quad r(t) = \langle 1, -\frac{3\sqrt{3}}{2} \rangle + t \langle 2, \sqrt{3} \rangle. \]

Further properties of the gradient vector.

Theorem
If \( f, g \) are differentiable scalar valued vector functions, \( g \neq 0 \), and \( k \in \mathbb{R} \) any constant, then holds,

1. \( \nabla (kf) = k (\nabla f) \);
2. \( \nabla (f \pm g) = \nabla f \pm \nabla g \);
3. \( \nabla (fg) = (\nabla f) g + f (\nabla g) \);
4. \( \nabla \left( \frac{f}{g} \right) = \frac{\nabla f \ g - f \ (\nabla g)}{g^2} \).
Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
- The tangent plane to the graph of a function.
- The linear approximation of a differentiable function.
- Bounds for the error of a linear approximation.
- The differential of a function.
  - Review: Scalar functions of one variable.
  - Scalar functions of more than one variable.

Review: Differentiable functions of two variables.

**Definition**

Given a function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) and an interior point \((x_0, y_0) \in D\), let \( L \) be the linear function

\[
L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).
\]

The function \( f \) is called **differentiable at** \((x_0, y_0)\) iff the function \( f \) is approximated by the linear function \( L \) near \((x_0, y_0)\), that is,

\[
f(x, y) = L(x, y) + \epsilon_1 (x - x_0) + \epsilon_2 (y - y_0)
\]

where the functions \( \epsilon_1 \) and \( \epsilon_2 \rightarrow 0 \) as \((x, y) \rightarrow (x_0, y_0)\).

**Theorem**

*If the partial derivatives \( f_x \) and \( f_y \) of a function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) are continuous in an open region \( R \subset D \), then \( f \) is differentiable in \( R \).*
Example
Show that the function $f(x, y) = x^2 + y^2$ is differentiable for all $(x, y) \in \mathbb{R}^2$. Furthermore, find the linear function $L$, mentioned in the definition of a differentiable function, at the point $(1, 2)$.

Solution: The partial derivatives of $f$ are given by $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, which are continuous functions. Therefore, the function $f$ is differentiable. The linear function $L$ at $(1, 2)$ is

$$L(x, y) = f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) + f(1, 2).$$

That is, we need three numbers to find the linear function $L$: $f_x(1, 2)$, $f_y(1, 2)$, and $f(1, 2)$. These numbers are:

$$f_x(1, 2) = 2, \quad f_y(1, 2) = 4, \quad f(1, 2) = 5.$$

Therefore, $L(x, y) = 2(x - 1) + 4(y - 2) + 5$. ◯
The tangent plane to the graph of a function.

Remark:
The function \( L(x, y) = 2(x - 1) + 4(y - 2) + 5 \) is a plane in \( \mathbb{R}^3 \). We usually write down the equation of a plane using the notation \( z = L(x, y) \), that is, \( z = 2(x - 1) + 4(y - 2) + 5 \), or equivalently

\[
2(x - 1) + 4(y - 2) - (z - 5) = 0.
\]

This is a plane passing through \( \tilde{P}_0 = (1, 2, 5) \) with normal vector \( n = \langle 2, 4, -1 \rangle \). Analogously, the function

\[
L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)
\]

is a plane in \( \mathbb{R}^3 \). Using the notation \( z = L(x, y) \) we obtain

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.
\]

This is a plane passing through \( \tilde{P}_0 = (x_0, y_0, f(x_0, y_0)) \) with normal vector \( n = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle \).

Theorem

The plane tangent to the graph of a differentiable function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) at the point \( (x_0, y_0) \) is given by

\[
L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).
\]

Proof

The plane contains the point \( \tilde{P}_0 = (x_0, y_0, f(x_0, y_0)) \). We only need to find its normal vector \( n \).
The tangent plane to the graph of a function.

The vector \( \mathbf{n} \) normal to the plane \( L(x, y) \) is a vector perpendicular to the surface \( z = f(x, y) \) at \( P_0 = (x_0, y_0) \).

This surface is the level surface \( F(x, y, z) = 0 \) of the function \( F(x, y, z) = f(x, y) - z \). A vector normal to this level surface is its gradient \( \nabla F \). That is, \( \nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle \).

Therefore, the normal to the tangent plane \( L(x, y) \) at the point \( P_0 \) is \( \mathbf{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle \). Recall that the plane contains the point \( \tilde{P}_0 = (x_0, y_0, f(x_0, y_0)) \). The equation for the plane is

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.
\]

The tangent plane to the graph of a function.

**Summary:** We have shown that the linear \( L \) given by

\[
L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)
\]

is the plane tangent to the graph of \( f \) at \( (x_0, y_0) \).

**Remark:** The graph of a differentiable function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is approximated by the tangent plane \( L \) at every point in \( D \).
The tangent plane to the graph of a function.

Example
Show that $f(x, y) = \arctan(x + 2y)$ is differentiable and find the plane tangent to $f(x, y)$ at $(1, 0)$.

Solution: The partial derivatives of $f$ are given by

$$f_x(x, y) = \frac{1}{1 + (x + 2y)^2}, \quad f_y(x, y) = \frac{2}{1 + (x + 2y)^2}.$$  

These functions are continuous in $\mathbb{R}^2$, so $f(x, y)$ is differentiable at every point in $\mathbb{R}^2$. The plane $L(x, y)$ at $(1, 0)$ is given by

$$L(x, y) = f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) + f(1, 0),$$

where $f(1, 0) = \arctan(1) = \pi/4$, $f_x(1, 0) = 1/2$, $f_y(1, 0) = 1$.

Then, $L(x, y) = \frac{1}{2}(x - 1) + y + \frac{\pi}{4}$.  

Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
- The tangent plane to the graph of a function.
- The linear approximation of a differentiable function.
- Bounds for the error of a linear approximation.
- The differential of a function.
  - Review: Scalar functions of one variable.
  - Scalar functions of more than one variable.
The linear approximation of a differentiable function.

Definition
The linear approximation of a differentiable function \( f: D \subset \mathbb{R}^2 \to \mathbb{R} \) at the point \((x_0, y_0) \in D\) is the plane

\[
L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).
\]

Example
Find the linear approximation of \( f = \sqrt{17 - x^2 - 4y^2} \) at \( (2, 1) \).
Solution: \( L(x, y) = f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) + f(2, 1) \).
We need three numbers: \( f(2, 1), f_x(2, 1), \) and \( f_y(2, 1). \)
These are: \( f(2, 1) = 3, f_x(2, 1) = -2/3, \) and \( f_y(2, 1) = -4/3. \)
Then the plane is given by \( L(x, y) = -\frac{2}{3}(x - 2) - \frac{4}{3}(y - 1) + 3. \)

Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
- The tangent plane to the graph of a function.
- The linear approximation of a differentiable function.
- **Bounds for the error of a linear approximation.**
- The differential of a function.
  - Review: Scalar functions of one variable.
  - Scalar functions of more than one variable.
Bounds for the error of a linear approximation.

**Theorem**
Assume that the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has first and second partial derivatives continuous on an open set containing a rectangular region $R \subset D$ centered at the point $(x_0, y_0)$.
If $M \in \mathbb{R}$ is the upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ in $R$, then the error $E(x, y) = f(x, y) - L(x, y)$ satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2,$$

where $L(x, y)$ is the linearization of $f$ at $(x_0, y_0)$, that is,

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

Bounds for the error of a linear approximation.

**Example**
Find an upper bound for the error in the linear approximation of $f(x, y) = x^2 + y^2$ at the point $(1, 2)$ over the rectangle $R = \{(x, y) \in \mathbb{R}^2 : |x - 1| < 0.1, |y - 2| < 0.1\}$

**Solution**: The second derivatives of $f$ are $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$.
Therefore, we can take $M = 2$.
Then the formula $|E(x, y)| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2$, implies

$$|E(x, y)| \leq (|x - 1| + |y - 2|)^2 < (0.1 + 0.1)^2 = 0.04,$$

that is $|E(x, y)| < 0.04$. Since $f(1, 2) = 5$, the percentage relative error $100 \frac{E(x, y)}{f(1, 2)}$ is bounded by 0.8% △
Review: Differential of functions of one variable.

Definition
The differential at $x_0 \in D$ of a differentiable function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ is the linear function

$$df(x) = L(x) - f(x_0).$$

Remark: The linear approximation of $f(x)$ at $x_0$ is the line given by $L(x) = f'(x_0)(x - x_0) + f(x_0)$.

Therefore

$$df(x) = f'(x_0)(x - x_0).$$

Denoting $dx = x - x_0$,

$$df = f'(x_0)\,dx.$$
Differential of functions of more than one variable.

Definition

The \textit{differential at} \((x_0, y_0) \in D\) of a differentiable function \(f : D \subset \mathbb{R}^2 \to \mathbb{R}\) is the linear function

\[ df(x, y) = L(x, y) - f(x_0, y_0). \]

Remark: The linear approximation of \(f(x, y)\) at \((x_0, y_0)\) is the plane \(L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)\).

Therefore \(df(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\).

Denoting \(dx = x - x_0\) and \(dy = (y - y_0)\) we obtain the usual expression

\[ df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy. \]

Therefore, \(df\) and \(L\) are similar concepts: \textit{The linear approximation of a differentiable function} \(f\).

Example

Compute the \(df\) of the function \(f(x, y) = \ln(1 + x^2 + y^2)\) at the point \((1, 1)\). Evaluate this \(df\) for \(dx = 0.1\), \(dy = 0.2\).

Solution: The differential of \(f\) at \((x_0, y_0)\) is given by

\[ df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy. \]

The partial derivatives \(f_x\) and \(f_y\) are given by

\[ f_x(x, y) = \frac{2x}{1 + x^2 + y^2}, \quad f_y(x, y) = \frac{2y}{1 + x^2 + y^2}. \]

Therefore, \(f_x(1, 1) = 2/3 = f_y(1, 1)\). Then \(df = \frac{2}{3}\, dx + \frac{2}{3}\, dy\).

Evaluating this differential at \(dx = 0.1\) and \(dy = 0.2\) we obtain

\[ df = \frac{2}{3} \cdot \frac{1}{10} + \frac{2}{3} \cdot \frac{2}{10} = \frac{2}{3} \cdot \frac{3}{10} \Rightarrow df = \frac{1}{5}. \]
Differential of functions of more than one variable.

Example

Use differentials to estimate the amount of aluminum needed to build a closed cylindrical can with internal diameter of 8 cm and height of 12 cm if the aluminum is 0.04 cm thick.

Solution:

The data of the problem is: \( h_0 = 12 \text{ cm}, \ r_0 = 4 \text{ cm}, \ dr = 0.04 \text{ cm} \) and \( dh = 0.08 \text{ cm} \).

The function to consider is the mass of the cylinder, \( M = \rho V \), where \( \rho = 2.7 \text{gr/cm}^3 \) is the aluminum density and \( V \) is the volume of the cylinder,

\[
V(r, h) = \pi r^2 h.
\]

The metal to build the can is given by

\[
\Delta M = \rho \left[ V(r + dr, h + dh) - V(r, h) \right], \quad \text{(recall } dh = 2dr. \text{)}
\]
Review: Local extrema for functions of one variable.

Recall: Main results on local extrema for \( f(x) \):

\[
\begin{array}{c|c|c|c}
\text{at} & f & f' & f'' \\
\hline
a & \text{max.} & 0 & < 0 \\
b & \text{infl.} & \neq 0 & \pm 0 \mp \\
c & \text{min.} & 0 & > 0 \\
d & \text{infl.} & = 0 & \pm 0 \mp \\
\end{array}
\]

Remarks: Assume that \( f \) is twice continuously differentiable.

- If \( x_0 \) is local maximum or minimum of \( f \), then \( f'(x_0) = 0 \).
- If \( f'(x_0) = 0 \) then \( x_0 \) is a critical point of \( f \), that is, \( x_0 \) is a maximum or a minimum or an inflection point.
- The second derivative test determines whether a critical point is a maximum, minimum or an inflection point.
Local and absolute extrema, saddle points (Sect. 14.7).

- Review: Local extrema for functions of one variable.
- **Definition of local extrema.**
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.

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**Definition**
A function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) has a **local maximum** at the point \((a, b) \in D\) iff holds that \( f(x, y) \leq f(a, b) \) for every point \((x, y)\) in a neighborhood of \((a, b)\).

A function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) has a **local minimum** at the point \((a, b) \in D\) iff holds that \( f(x, y) \geq f(a, b) \) for every point \((x, y)\) in a neighborhood of \((a, b)\).
Definition of local extrema for functions of two variables.

**Definition**
A differentiable function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) has a *saddle point* at an interior point \((a, b) \in D\) iff in every open disk in \(D\) centered at \((a, b)\) there always exist points \((x, y)\) where \(f(x, y) > f(a, b)\) and other points \((x, y)\) where \(f(x, y) < f(a, b)\).

Local and absolute extrema, saddle points (Sect. 14.7).

▶ Review: Local extrema for functions of one variable.
▶ Definition of local extrema.
▶ **Characterization of local extrema.**
  ▶ First derivative test.
  ▶ Second derivative test.
▶ Absolute extrema of a function in a domain.
Characterization of local extrema. 
First derivative test.

**Theorem**
If a differentiable function \( f \) has a local maximum or minimum at \((a, b)\) then holds \( (\nabla f)\big|_{(a,b)} = \langle 0, 0 \rangle \).

**Remark:** The tangent plane at a local extremum is horizontal, since its normal vector is \( \mathbf{n} = \langle f_x, f_y, -1 \rangle = \langle 0, 0, -1 \rangle \).

**Definition**
The interior point \((a, b)\) \(\in D\) of a differentiable function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is a critical point of \( f \) iff \( (\nabla f)\big|_{(a,b)} = \langle 0, 0 \rangle \).

**Remark:**
Critical points include local maxima, local minima, and saddle points.

---

**Example**
Find the critical points of the function \( f(x, y) = -x^2 - y^2 \)

**Solution:** The critical points are the points where \( \nabla f \) vanishes. Since \( \nabla f = \langle -2x, -2y \rangle \), the only solution to \( \nabla f = \langle 0, 0 \rangle \) is \( x = 0, y = 0 \). That is, \( (a, b) = (0, 0) \).  

**Remark:** Since \( f(x, y) \leq 0 \) for all \( (x, y) \in \mathbb{R}^2 \) and \( f(0, 0) = 0 \), then the point \((0, 0)\) must be a local maximum of \( f \).

**Example**
Find the critical points of the function \( f(x, y) = x^2 - y^2 \)

**Solution:** Since \( \nabla f = \langle 2x, -2y \rangle \), the only solution to \( \nabla f = \langle 0, 0 \rangle \) is \( x = 0, y = 0 \). That is, we again obtain \( (a, b) = (0, 0) \).
Characterization of local extrema.
Second derivative test.

Theorem
Let \((a, b)\) be a critical point of \(f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}\), that is, \(\left(\nabla f\right)\big|_{(a,b)} = (0, 0)\). Assume that \(f\) has continuous second derivatives in an open disk in \(D\) with center in \((a, b)\) and denote

\[ D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2. \]

Then, the following statements hold:
- If \(D > 0\) and \(f_{xx}(a,b) > 0\), then \(f(a,b)\) is a local minimum.
- If \(D > 0\) and \(f_{xx}(a,b) < 0\), then \(f(a,b)\) is a local maximum.
- If \(D < 0\), then \(f(a,b)\) is a saddle point.
- If \(D = 0\) the test is inconclusive.

Notation: The number \(D\) is called the discriminant of \(f\) at \((a,b)\).

Characterization of local extrema.
Second derivative test.

Example
Find the local extrema of \(f(x, y) = y^2 - x^2\) and determine whether they are local maximum, minimum, or saddle points.

Solution: We first find the critical points:

\[ \nabla f = (-2x, 2y) \quad \Rightarrow \quad \left(\nabla f\right)\big|_{(a,b)} = (0, 0) \quad \text{iff} \quad (a, b) = (0, 0). \]

The only critical point is \((a, b) = (0, 0)\).
We need to compute \(D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2\).
Since \(f_{xx}(0,0) = -2\), \(f_{yy}(0,0) = 2\), and \(f_{xy}(0,0) = 0\), we get

\[ D = (-2)(2) = -4 < 0 \quad \Rightarrow \quad \text{saddle point at} \ (0,0). \]
Example

Is the point \((a, b) = (0, 0)\) a local extrema of \(f(x, y) = y^2x^2\)?

Solution: We first verify that \((0, 0)\) is a critical point of \(f\):

\[
\nabla f(x, y) = \langle 2xy^2, 2yx^2 \rangle, \quad \Rightarrow \quad (\nabla f)(0, 0) = \langle 0, 0 \rangle,
\]

therefore, \((0, 0)\) is a critical point.

Remark: The whole axes \(x = 0\) and \(y = 0\) are critical points of \(f\).

We need to compute \(D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2\).

Since \(f_{xx}(x, y) = 2y^2\), \(f_{yy}(x, y) = 2x^2\), and \(f_{xy}(x, y) = 4xy\),

we obtain \(f_{xx}(0, 0) = 0\), \(f_{yy}(0, 0) = 0\), and \(f_{xy}(0, 0) = 0\),

hence \(D = 0\) and the test is inconclusive.

---

Example

Is the point \((a, b) = (0, 0)\) a local extrema of \(f(x, y) = y^2x^2\)?

Solution: From the graph of \(f = x^2y^2\) is simple to see that \((0, 0)\) is a local minimum: (also a global minimum.)
Local and absolute extrema, saddle points (Sect. 14.7).

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.

Absolute extrema of a function in a domain.

**Definition**
A function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) has an **absolute maximum** at the point \((a, b) \in D\) iff \( f(x, y) \leq f(a, b) \) for all \((x, y) \in D\).
A function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) has an **absolute minimum** at the point \((a, b) \in D\) iff \( f(x, y) \geq f(a, b) \) for all \((x, y) \in D\).

**Remark:** Local extrema need not be the absolute extrema.

**Remark:** Absolute extrema may not be defined on open intervals.
Review: Functions of one variable.

Theorem

*Every continuous functions* $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}, \text{ with } a < b \in \mathbb{R}$ *always has absolute extrema.*

Recall:
- Intervals $[a, b]$ are bounded and closed sets in $\mathbb{R}$.
- The set $[a, b]$ is closed, since the boundary points belong to the set, and it is bounded, since it does not extend to infinity.

Recall: On open and closed sets in $\mathbb{R}^n$.

Definition

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in $S$ is an interior point. The set $S$ is called *closed* iff $S$ contains its boundary. A set $S$ is called *bounded* iff $S$ is contained in ball, otherwise $S$ is called *unbounded*.

Theorem

*If $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous in a closed and bounded set $D$, then $f$ has an absolute maximum and an absolute minimum in $D.*
Absolute extrema on closed and bounded sets.

**Problem:**
Find the absolute extrema of a function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ in a closed and bounded set $D$.

**Solution:**
(1) Find every critical point of $f$ in the interior of $D$ and evaluate $f$ at these points.
(2) Find the boundary points of $D$ where $f$ has local extrema, and evaluate $f$ at these points.
(3) Look at the list of values for $f$ found in the previous two steps.
   If $f(x_0, y_0)$ is the biggest (smallest) value of $f$ in the list above, then $(x_0, y_0)$ is the absolute maximum (minimum) of $f$ in $D$.

---

**Example**
Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.

**Solution:**
(1) We find all critical points in the interior of the domain:
   \[
   \nabla f = \langle (y - 1), (x + 2) \rangle = \langle 0, 0 \rangle \implies (x_0, y_0) = (-2, 1).
   \]
Since $(-2, 1)$ does not belong to the domain, we discard it.

(2) Three segments form the boundary of $D$:
   **Boundary I:** The segment $y = 0$, $x \in [1, 5]$. We select the end points $(1, 0)$, $(5, 0)$, and we record: $f(1, 0) = 2$ and $f(5, 0) = -2$.
   We look for critical point on the interior of Boundary I: Since $g(x) = f(x, 0) = 3 - x$, so $g' = -1 \neq 0$. No critical points in the interior of Boundary I.
Absolute extrema on closed and bounded sets.

Example
Find the absolute extrema of the function 
\[ f(x, y) = 3 + xy - x + 2y \] on the closed domain given in the Figure.

Solution: Boundary II: The segment \( x = 1, \ y \in [0, 4] \). We select the end point \((1, 4)\) and we record: \( f(1, 4) = 14 \).
We look for critical point on the interior of Boundary II: Since 
\[ g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y, \] so \( g' = 3 \neq 0 \). No critical points in the interior of Boundary II.

Boundary III: The segment \( y = -x + 5, \ x \in [1, 5] \).
We look for critical point on the interior of Boundary III: Since 
\[ g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5). \] We obtain 
\[ g(x) = -x^2 + 2x + 13, \] hence \( g'(x) = -2x + 2 = 0 \) implies \( x = 1 \).
So, \( y = 4 \), and we selected the point \((1, 4)\), which was already in our list. No critical points in the interior of Boundary III.

Absolute extrema on closed and bounded sets.

Example
Find the absolute extrema of the function 
\[ f(x, y) = 3 + xy - x + 2y \] on the closed domain given in the Figure.

Solution:
(3) Our list of values is:
\[ f(1, 0) = 2 \quad f(1, 4) = 14 \quad f(5, 0) = -2. \]
We conclude:
- Absolute maximum at \((1, 4)\),
- Absolute minimum at \((5, 0)\).
Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

Solution: This problem can be solved by finding the local maximum of an appropriate function $f$.

The function $f$ is obtained as follows: Recall the functions volume and area of a rectangular box with vertex at $(0, 0, 0)$ and sides $x, y$ and $z$:

$$V(x, y, z) = xyz, \quad A(x, y, z) = 2xy + 2xz + 2yz.$$  

Since $A(x, y, z) = A_0$, we obtain $z = \frac{A_0 - 2xy}{2(x + y)}$, that is

$$f(x, y) = \frac{A_0xy - 2x^2y^2}{2(x + y)}.$$

Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

Solution:
We must find the critical points of $f(x, y) = \frac{A_0xy - 2x^2y^2}{2(x + y)}$.

$$f_x = \frac{2A_0y^2 - 4x^2y^2 - 8xy^3}{4(x + y)^2}, \quad f_y = \frac{2A_0x^2 - 4x^2y^2 - 8yx^3}{4(x + y)^2}.$$  

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0, y \neq 0$ imply

$$A_0 = 2x^2 + 4xy, \quad A_0 = 2y^2 + 4xy, \quad \Rightarrow \quad x = y.$$  

Recall $z = \frac{A_0 - 2xy}{2(x + y)}$, so, $z = \frac{A_0 - 2x^2}{4x} = y$. Therefore, $x_0 = y_0 = z_0 = \sqrt{A_0/6}$. 