Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives and continuity.

Recall: The following result holds for single variable functions.

**Theorem**

*If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then $f$ is continuous.*

**Proof.**

$$
\lim_{h \to 0} [f(x + h) - f(x)] = \lim_{h \to 0} \left[ \frac{f(x + h) - f(x)}{h} \right] h,
$$

$$
= f'(x) \lim_{h \to 0} h
$$

$$
= 0.
$$

That is, $\lim_{h \to 0} f(x + h) = f(x)$, so $f$ is continuous.

However, the claim “If $f_x(x, y)$ and $f_y(x, y)$ exist, then $f(x, y)$ is continuous” is false.
Partial derivatives and continuity.

Theorem

If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then $f$ is continuous.

Remark:

- This Theorem is not true for the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$.
- There exist functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist but $f$ is not continuous at $(x_0, y_0)$.

Remark: This is a bad property for a differentiable function.

Partial derivatives and continuity.

Remark: Here is a discontinuous function at $(0, 0)$ having partial derivatives at $(0, 0)$.

Example

(a) Show that $f$ is not continuous at $(0, 0)$, where

$$f(x, y) = \begin{cases} 
\frac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0), \\
0 & (x, y) = (0, 0).
\end{cases}$$

(b) Find $f_x(0, 0)$ and $f_y(0, 0)$.

Solution:

(a) Choosing the path $x = 0$ we see that $f(0, y) = 0$, so $\lim_{y \to 0} f(0, y) = 0$. Choosing the path $x = y$ we see that $f(x, x) = 2x^2/2x^2 = 1$, so $\lim_{x \to 0} f(x, x) = 1$. The Two-Path Theorem implies that $\lim_{(x, y) \to (0,0)} f(x, y)$ does not exist.
Example

(a) Show that $f$ is not continuous at $(0,0)$, where

$$f(x, y) = \begin{cases} 
  \frac{2xy}{x^2 + y^2} & (x, y) \neq (0,0), \\
  0 & (x, y) = (0,0).
\end{cases}$$

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

Solution:
(b) The partial derivatives are defined at $(0,0)$.

$$f_x(0,0) = \lim_{h \to 0} \frac{1}{h} [f(0 + h, 0) - f(0,0)] = \lim_{h \to 0} \frac{1}{h} [0 - 0] = 0.$$ 

$$f_y(0,0) = \lim_{h \to 0} \frac{1}{h} [f(0,0 + h) - f(0,0)] = \lim_{h \to 0} \frac{1}{h} [0 - 0] = 0.$$ 

Therefore, $f_x(0,0) = f_y(0,0) = 0$. △

Partial derivatives and differentiability (Sect. 14.3).

► Partial derivatives and continuity.
► **Differentiable functions** $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
► Differentiability and continuity.
► A primer on differential equations.
Differentiable functions \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \).

Recall: A differentiable function \( f : \mathbb{R} \rightarrow \mathbb{R} \) at \( x_0 \) must be approximated by a line \( L(x) \) containing \( x_0 \) with slope \( f'(x_0) \).

The equation of the tangent line is

\[
L(x) = f'(x_0)(x - x_0) + f(x_0).
\]

The function \( f \) is approximated by the line \( L \) near \( x_0 \) means

\[
f(x) = L(x) + \epsilon_1(x - x_0)
\]

with \( \epsilon_1(x) \rightarrow 0 \) as \( x \rightarrow x_0 \).

The graph of a differentiable function \( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \) is approximated by a line at every point in \( D \).

Remark: The idea to define differentiable functions:
The graph of a differentiable function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is approximated by a plane at every point in \( D \).

We will show next week that the equation of the plane \( L \) is

\[
L(x,y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).
\]
Definition
Given a function \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) and an interior point \((x_0, y_0)\) in \( D \), let \( L \) be the plane given by

\[
L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).
\]

The function \( f \) is called **differentiable at** \((x_0, y_0)\) iff the function \( f \) is approximated by the plane \( L \) near \((x_0, y_0)\), that is,

\[
f(x, y) = L(x, y) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0)
\]

where the functions \( \epsilon_1 \) and \( \epsilon_2 \to 0 \) as \((x, y) \to (x_0, y_0)\).

The function \( f \) is **differentiable** iff \( f \) is differentiable at every interior point of \( D \).

Remark: Recalling that the equation for the plane \( L \) is

\[
L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0),
\]

an equivalent expression for \( f \) being differentiable,

\[
f(x, y) = L(x, y) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0),
\]

is the following: Denote \( z = f(x, y) \) and \( z_0 = f(x_0, y_0) \), and introduce the increments

\[
\Delta z = (z - z_0), \quad \Delta y = (y - y_0), \quad \Delta x = (x - x_0);
\]

then, the equation above is

\[
\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y.
\]

(Equation used in the textbook to define a differentiable function.)
Recall: The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in $D$.

Remark: A simple sufficient condition on a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ guarantees that $f$ is differentiable:

**Theorem**

If the partial derivatives $f_x$ and $f_y$ of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then $f$ is differentiable in $R$.

**Theorem**

If a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable, then $f$ is continuous.
Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
- Differentiable functions \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).
- Differentiability and continuity.
- **A primer on differential equations.**

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A primer on differential equations.

**Remark:** A **differential equation** is an equation where the unknown is a function and the function together with its derivatives appear in the equation.

**Example**

Given a constant \( k \in \mathbb{R} \), find all solutions \( f : \mathbb{R} \to \mathbb{R} \) to the differential equation

\[
f'(x) = kf(x).
\]

**Solution:** Multiply the equation above \( f'(x) - kf(x) = 0 \) by \( e^{-kx} \), that is, \( f'(x)e^{-kx} - f(x)ke^{-kx} = 0 \).

The left-hand side is a total derivative, \( [f(x)e^{-kx}]' = 0 \).

The solution of the equation above is \( f(x)e^{-kx} = c \), with \( c \in \mathbb{R} \).

Therefore, \( f(x) = c e^{kx} \). ⬤
A primer on differential equations.

There are three differential equations for functions $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3, 4$, that appear in several physical applications.

- The Laplace equation: (Gravitation, electrostatics.)
  $$\partial_x^2 f + \partial_y^2 f + \partial_z^2 f = 0.$$

- The Heat equation: (Heat propagation, diffusion.)
  $$\partial_t f = k(\partial_x^2 f + \partial_y^2 f + \partial_z^2 f).$$

- The Wave equation: (Light, sound, gravitation.)
  $$\partial_t^2 f = v(\partial_x^2 f + \partial_y^2 f + \partial_z^2 f).$$

Example

Verify that $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation: $f_{xx} + f_{yy} + f_{zz} = 0$.

Solution: Recall: $f_x = -x/(x^2 + y^2 + z^2)^{3/2}$. Then,

$$f_{xx} = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{2} \frac{2x^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $f_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$. Analogously, $f_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$, and $f_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$. Then,

$$f_{xx} + f_{yy} + f_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0.$$

We conclude that $f_{xx} + f_{yy} + f_{zz} = 0$. ◁
Example
Verify that the function $T(t, x) = e^{-4t} \sin(2x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.

Solution: We first compute $T_t$,

$$T_t = -4e^{-t} \sin(2x).$$

Now compute $T_{xx}$,

$$T_x = 2e^{-t} \cos(2x) \Rightarrow T_{xx} = -4e^{-t} \sin(2x)$$

Therefore $T_t = T_{xx}$. 

Example
Verify that the function $f(t, x) = (vt - x)^3$, with $v \in \mathbb{R}$, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

Solution: We first compute $f_{tt}$,

$$f_t = 3v(vt - x)^2 \Rightarrow f_{tt} = 6v^2(vt - x).$$

Now compute $f_{xx}$,

$$f_x = -3(vt - x)^2 \Rightarrow f_{xx} = 6(vt - x).$$

Therefore $f_{tt} = v^2 f_{xx}$. 

A primer on differential equations.
A primer on differential equations.

Example
Verify that every function \( f(t, x) = u(vt - x) \), with \( v \in \mathbb{R} \) and \( u : \mathbb{R} \to \mathbb{R} \) twice continuously differentiable, satisfies the one-space dimensional wave equation \( f_{tt} = v^2 f_{xx} \).

Solution: We first compute \( f_{tt} \),
\[
f_t = v \ u'(vt - x) \quad \Rightarrow \quad f_{tt} = v^2 \ u''(vt - x).
\]
Now compute \( f_{xx} \),
\[
f_x = -u'(vt - x)^2 \quad \Rightarrow \quad f_{xx} = u''(vt - x).
\]
Therefore \( f_{tt} = v^2 f_{xx} \).  

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for \( f : D \subset \mathbb{R} \to \mathbb{R} \).
  - The chain rule for change of coordinates in a line.
- Functions of two variables, \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.
Review: The chain rule for \( f : D \subset \mathbb{R} \to \mathbb{R} \).

The chain rule for change of coordinates in a line.

**Theorem**

*If the functions \( f : [x_0, x_1] \to \mathbb{R} \) and \( x : [t_0, t_1] \to [x_0, x_1] \) are differentiable, then the function \( \hat{f} : [t_0, t_1] \to \mathbb{R} \) given by the composition \( \hat{f}(t) = f(x(t)) \) is differentiable and

\[
\frac{d\hat{f}}{dt}(t) = \frac{df}{dx}(x(t)) \frac{dx}{dt}(t).
\]

**Notation:**

The equation above is usually written as \( \frac{d\hat{f}}{dt} = \frac{df}{dx} \frac{dx}{dt} \).

Alternative notations are \( \hat{f}'(t) = f'(x(t)) x'(t) \) and \( \hat{f}' = f' x' \).

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Review: The chain rule for \( f : D \subset \mathbb{R} \to \mathbb{R} \).

**Example**

The volume \( V \) of a gas in a balloon depends on the temperature \( F \) in Fahrenheit as follows: \( V(F) = k F^2 \). Let \( F(C) = (9/5)C + 32 \) be the temperature in Fahrenheit corresponding to \( C \) in Celsius. Find \( \hat{V}(C) = V(F(C)) \) and \( \hat{V}'(C) \).

**Solution:**

The function \( \hat{V} \) is the composition \( \hat{V}(C) = k \left( \frac{9}{5} C + 32 \right)^2 \).

Which could also be written as

\[
\hat{V}(C) = k \frac{81}{25} C^2 + 64k \frac{9}{5} C + k(32)^2.
\]

The formula \( \frac{d\hat{V}}{dC} = \frac{dV}{dF} \frac{dF}{dC} \) implies \( \hat{V}'(C) = 2k \left( \frac{9}{5} C + 32 \right) \frac{9}{5} \). □
The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for \( f : D \subset \mathbb{R} \to \mathbb{R} \).
  - The chain rule for change of coordinates in a line.
- Functions of two variables, \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Functions of two variables, \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).

The chain rule for functions defined on a curve in a plane.

**Theorem**

If the functions \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) and \( r : \mathbb{R} \to D \subset \mathbb{R}^2 \) are differentiable, with \( r(t) = (x(t), y(t)) \), then the function \( \hat{f} : \mathbb{R} \to \mathbb{R} \) given by the composition \( \hat{f}(t) = f(r(t)) \) is differentiable and holds

\[
\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}(r(t)) \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(r(t)) \frac{dy}{dt}(t).
\]

**Notation:**

The equation above is usually written as

\[
\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

An alternative notation is \( \hat{f}' = (\partial_x f) x' + (\partial_y f) y' \).
Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

Example
Evaluate the function $f(x, y) = x^2 + 2y^3$, along the curve $r(t) = \langle x(t), y(t) \rangle = \langle \sin(t), \cos(2t) \rangle$. Furthermore, compute the derivative of $f$ along that curve.

Solution: The function $f$ along the curve $r(t)$ is denoted as $\hat{f}(t) = f(x(t), y(t))$. The result is $\hat{f}(t) = \sin^2(t) + 2\cos^3(2t)$.

The derivative of $f$ along the curve $r$ is $\hat{f}'$. The result is

$$\hat{f}'(t) = 2x(t)x'(t) + 6(y(t))^2 y'(t),$$

$$= 2x(t)\cos(t) - 12(y(t))^2 \sin(2t)$$

We conclude: $\hat{f}'(t) = 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t)$. ◢

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Theorem
*If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ and the change of coordinate functions $x, y : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t, s)$ and $y(t, s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t, s) = f(x(t, s), y(t, s))$ is differentiable and holds

$$\hat{f}_t = f_x x_t + f_y y_t$$

$$\hat{f}_s = f_x x_s + f_y y_s.$$*

Remark:
We denote by $f(x, y)$ are the function values in the coordinates $(x, y)$, while we denote by $\hat{f}(t, s)$ are the function values in the coordinates $(t, s)$. 
Functions of two variables, $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates $(x, y)$, find $\hat{f}(r, \theta)$ in polar coordinates $(r, \theta)$. Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: The polar coordinates $(r, \theta)$ are related to Cartesian coordinates $(x, y)$ by the formula

$$x(r, \theta) = r \cos(\theta), \quad y(r, \theta) = r \sin(\theta).$$

The function $\hat{f}(r, \theta) = f(x(r, \theta), y(r, \theta))$ is simple to compute,

$$\hat{f}(r, \theta) = r^2 \cos^2(\theta) + 3r^2 \sin^2(\theta).$$

Recall: $x(r, \theta) = r \cos(\theta)$ and $y(r, \theta) = r \sin(\theta)$.

Compute the derivatives of $\hat{f}(r, \theta) = r^2 \cos^2(\theta) + 3r^2 \sin^2(\theta)$.

$$\hat{f}_r = f_x x_r + f_y y_r = 2x \cos(\theta) + 6y \sin(\theta).$$

we obtain $\hat{f}_r = 2r \cos^2(\theta) + 6r \sin^2(\theta)$. Analogously,

$$\hat{f}_\theta = f_x x_\theta + f_y y_\theta = -2xr \sin(\theta) + 6yr \cos(\theta).$$

we obtain $\hat{f}_\theta = -2r^2 \cos(\theta) \sin(\theta) + 6r^2 \cos(\theta) \sin(\theta)$.
The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for \( f : D \subset \mathbb{R} \to \mathbb{R} \).
  - The chain rule for change of coordinates in a line.
- Functions of two variables, \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- **Functions of three variables**, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).

The chain rule for functions defined on a curve in space.

**Theorem**

*If the functions \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \) and \( r : \mathbb{R} \to D \subset \mathbb{R}^3 \) are differentiable, with \( r(t) = (x(t), y(t), z(t)) \), then the function \( \hat{f} : \mathbb{R} \to \mathbb{R} \) given by the composition \( \hat{f}(t) = f(r(t)) \) is differentiable and holds*

\[
\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.
\]

**Notation:**

The equation above is usually written as

\[
\hat{f}' = f_x x' + f_y y' + f_z z'.
\]
Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).

The chain rule for functions defined on a curve in space.

**Example**

Find the derivative of \( f = x^2 + y^3 + z^4 \) along the curve \( r(t) = \langle \cos(t), \sin(t), 3t \rangle \).

**Solution:** We first compute \( \hat{f}(t) = f(x(t), y(t), z(t)) \), that is,
\[
\hat{f}(t) = \cos^2(t) + \sin^3(t) + 81 \ t^4.
\]

The derivative of \( f \) along the curve \( r \) is the derivative of \( \hat{f} \), that is,
\[
\hat{f}' = f_x x' + f_y y' + f_z z' = -2x \sin(t) + 3y^2 \cos(t) + 4z^3(3).
\]

We obtain \( \hat{f}' = -2 \cos(t) \sin(t) + 3 \sin^2(t) \cos(t) + 4(3)(3^3)t^3 \). \( \triangleq \)

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Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).

The chain rule for functions defined on surfaces in space.

**Theorem**

If the functions \( f : \mathbb{R}^3 \to \mathbb{R} \) and the surface given by functions \( x, y, z : \mathbb{R}^2 \to \mathbb{R} \) are differentiable, with \( x(t, s) \) and \( y(t, s) \), and \( z(t, s) \), then the function \( \hat{f} : \mathbb{R}^2 \to \mathbb{R} \) given by the composition \( \hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s)) \) is differentiable and holds

\[
\hat{f}_t = f_x x_t + f_y y_t + f_z z_t, \\
\hat{f}_s = f_x x_s + f_y y_s + f_z z_s.
\]

**Remark:**

We denote by \( f(x, y, z) \) the function values in the coordinates \( (x, y, z) \), while we denote by \( \hat{f}(t, s) \) the function values at the surface point with coordinates \( (t, s) \).
Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).

The chain rule for functions defined on surfaces in space.

**Example**

Given the function \( f(x, y) = x^2 + 3y^2 + 2z^2 \), in Cartesian coordinates \((x, y)\), find \( \hat{f}(t, s) \), the values of \( f \) and its derivatives on the surface given by \( x(t, s) = t + s, y(t, s) = t^2 - s^2, \) \( z(t, s) = t - s. \)

**Solution:** The function \( \hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s)) \) is simple to compute: \( \hat{f}(t, s) = (t + s)^2 + 3(t^2 - s^2)^2 + 2(t - s)^2 \).

The derivatives of \( f \) along the surface \( x(t, s), y(t, s) \) and \( z(t, s) \) are given by \( \hat{f}_t \) and \( \hat{f}_s \); which are given by

\[
\hat{f}_t = f_x x_t + f_y y_t + f_z z_t
\]
\[
\hat{f}_s = f_x x_s + f_y y_s + f_z z_s.
\]

We obtain \( \hat{f}_t = 2(t + s) + 6(t^2 - s^2)(2t) + 4(t - s) \), and \( \hat{f}_s = 2(t + s) + 6(t^2 - s^2)(-2s) - 4(t - s). \)

\( \triangleq \)

Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).

The chain rule for change of coordinates in space.

**Theorem**

*If the functions \( f : \mathbb{R}^3 \to \mathbb{R} \) and the change of coordinate functions \( x, y, z : \mathbb{R}^3 \to \mathbb{R} \) are differentiable, with \( x(t, s, r), y(t, s, r), \) and \( z(t, s, r) \), then the function \( \hat{f} : \mathbb{R}^3 \to \mathbb{R} \) given by the composition \( \hat{f}(t, s, r) = f(x(t, s, r), y(t, s, r), z(t, s, r)) \) is differentiable and*

\[
\hat{f}_t = f_x x_t + f_y y_t + f_z z_t
\]
\[
\hat{f}_s = f_x x_s + f_y y_s + f_z z_s
\]
\[
\hat{f}_r = f_x x_r + f_y y_r + f_z z_r.
\]

**Remark:**

We denote by \( f(x, y, z) \) the function values in the coordinates \((x, y, z)\), while we denote by \( \hat{f}(t, s, r) \) the function values in the coordinates \((t, s, r)\).
Functions of three variables, \( f : D \subset \mathbb{R}^3 \rightarrow \mathbb{R} \).

The chain rule for change of coordinates in space.

Example

Given the function \( f(x, y, z) = x^2 + 3y^2 + z^2 \), in Cartesian coordinates \((x, y, z)\), find \( \hat{f}(r, \theta, \phi) \) and its derivatives in spherical coordinates \((r, \theta, \phi)\), where

\[
x = r \cos(\phi) \sin(\theta), \quad y = r \sin(\phi) \sin(\theta), \quad z = r \cos(\theta).
\]

Solution: We first compute the function

\[
\hat{f}(r, \theta, \phi) = f(x(r, \theta, \phi), y(r, \theta, \phi), z(r, \theta, \phi)),
\]

\[
\hat{f} = r^2 \cos^2(\phi) \sin^2(\theta) + 3r^2 \sin^2(\phi) \sin^2(\theta) + r^2 \cos^2(\theta)
\]

\[
= r^2 \sin^2(\theta) + 2r^2 \sin^2(\phi) \sin^2(\theta) + r^2 \cos^2(\theta)
\]

so we obtain

\[
\hat{f} = r^2 + 2r^2 \sin^2(\phi) \sin^2(\theta).
\]
The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for \( f : D \subset \mathbb{R} \to \mathbb{R} \).
  - The chain rule for change of coordinates in a line.
- Functions of two variables, \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

A formula for implicit differentiation.

**Theorem**

*Assume that the differentiable function with values \( F(x, y) \) defines implicitly a function with values \( y(x) \) by the equation \( F(x, y) = 0 \). If the function \( F_y \neq 0 \), then \( y \) is differentiable and*

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}.
\]

**Proof.**

*Since \( \hat{F}(x) = F(x, y(x)) = 0 \), then \( 0 = \frac{d\hat{F}}{dx} = F_x + F_y y' \).

We conclude that \( y' = -\frac{F_x}{F_y} \). \qed
A formula for implicit differentiation.

Example
Find the derivative of function \( y : \mathbb{R} \rightarrow \mathbb{R} \) defined implicitly by the equation \( F(x, y) = 0 \), where \( F(x, y) = x e^y + \cos(x - y) \).

Solution:
The partial derivatives of function \( F \) are

\[
F_x = e^y - \sin(x - y), \quad F_y = x e^y + \sin(x - y).
\]

Therefore,

\[
y'(x) = \frac{\sin(x - y) - e^y}{x e^y + \sin(x - y)}.
\]

\( \triangleleft \)

A formula for implicit differentiation.

Example
Find the derivative of function \( y : \mathbb{R} \rightarrow \mathbb{R} \) defined implicitly by the equation \( F(x, y) = 0 \), where \( F(x, y) = x e^y + \cos(x - y) \).

Solution: (Old method.)
Since \( F(x, y(x)) = x e^y + \cos(x - y) = 0 \), then

\[
e^y + x y' e^y - \sin(x - y) - \sin(x - y) (-y') = 0.
\]

Reordering terms,

\[
y' [x e^y + \sin(x - y)] = \sin(x - y) - e^y.
\]

We conclude that: \( y'(x) = \frac{\sin(x - y) - e^y}{x e^y + \sin(x - y)} \).  \( \triangleleft \)