Review for Exam 1.

- Sections 12.1-12.6.
- 50 minutes.
- 5 or 6 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Example
Consider the vectors \( \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{w} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \).

1. Compute \( \mathbf{v} \cdot \mathbf{w} \).

   Solution:
   
   \[
   \mathbf{v} \cdot \mathbf{w} = (2, -2, 1) \cdot (1, 2, -1) = 2 - 4 - 1 \Rightarrow \mathbf{v} \cdot \mathbf{w} = -3.
   \]

2. Find the cosine of the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

   Solution:
   
   \[
   |\mathbf{v}| = \sqrt{4 + 4 + 1} = 3, \quad |\mathbf{w}| = \sqrt{1 + 4 + 1} = \sqrt{6}.
   \]
   
   \[
   \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|} = \frac{-3}{3\sqrt{6}} \Rightarrow \cos(\theta) = -\frac{1}{\sqrt{6}}.
   \]
Example

1. Find a unit vector in the direction of \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).
   
   Solution:
   
   \[
   \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad |\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6},
   \]
   
   \[
   \mathbf{u} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle.
   \]

2. Find \(|\mathbf{u} - 2\mathbf{v}|\), where \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \), \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).
   
   Solution: First: \( \mathbf{u} - 2\mathbf{v} = \langle 1, 6, -1 \rangle \). Then,
   
   \[
   |\mathbf{u} - 2\mathbf{v}| = \sqrt{1 + 36 + 1} \quad \Rightarrow \quad |\mathbf{u} - 2\mathbf{v}| = \sqrt{38}.
   \]

Example

Find a unit vector \( \mathbf{u} \) normal to both \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \).

Solution:

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
6 & 2 & -3 \\
-2 & 2 & 1
\end{vmatrix} = (2 + 6)\mathbf{i} - (6 - 6)\mathbf{j} + (12 + 4)\mathbf{k} = \langle 8, 0, 16 \rangle.
\]

Since we look for a unit vector, the calculation is simpler with \( \langle 1, 0, 2 \rangle \) instead of \( \langle 8, 0, 16 \rangle \).

\[
\mathbf{u} = \frac{\langle 1, 0, 2 \rangle}{|\langle 1, 0, 2 \rangle|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{5}} \langle 1, 0, 2 \rangle.
\]
Example
Find the area of the parallelogram formed by \( \mathbf{v} \) and \( \mathbf{w} \) above.

Solution:
Since \( \mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle \), then

\[
A = |\mathbf{v} \times \mathbf{w}| = |\langle 8, 0, 16 \rangle| = \sqrt{8^2 + 16^2} = \sqrt{8^2(1 + 4)}.
\]

\[A = 8\sqrt{5}.
\]

Example
Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \).
We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle = \langle 9, 8, -4 \rangle.
\]

We obtain \( \mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle \). The triple product is

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.
\]

Since \( V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \), we obtain \( V = 82 \).
**Example**

Does the line given by $\mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t$ intersects the plane given by $2x + y - z = 1$? If the answer is yes, then find the intersection point.

**Solution:** The line with parametric equation

$$x = t, \quad y = 1 + 2t, \quad z = 1 + 3t,$$

intersect the plane $2x + y - z = 1$ iff there is a solution $t$ for the equation

$$2t + (1 + 2t) - (1 + 3t) = 1.$$

There is a solution given by $t = 1$. Therefore, the point of intersection has coordinates $x = 1$, $y = 3$, $z = 4$, then

$$P = (1, 3, 4).$$

\[\triangleright\]

**Example**

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

**Solution:**

The vector equation of the line is

$$\mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t.$$

A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$. The point $P_0 = (1, 2, 3)$ is in the plane. A second point in the plane is any point in the line, for example $P_1$ corresponding to the terminal point of $\mathbf{r}(0) = \langle -2, 0, -1 \rangle$.

Then a second vector tangent to the plane is $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$. 
Example
Find the equation for the plane that contains the point \( P_0 = (1, 2, 3) \) and the line \( x = -2 + t, \ y = t, \ z = -1 + 2t \).

Solution:
The vector equation of the line is 
\[ \mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t, \]
and a second vector tangent to the plane is 
\[ \overrightarrow{P_1 P_0} = \langle 3, 2, 4 \rangle. \]

Then, a normal to the plane is given by
\[ \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle 4 - 4, -(4 - 6), (2 - 3) \rangle \quad \Rightarrow \quad \mathbf{n} = \langle 0, 2, -1 \rangle. \]

So, the equation of the plane is
\[ 0(x - 1) + 2(y - 2) - (z - 3) = 0, \quad \Rightarrow \quad 2y - z = 1. \]

Example
Find an equation for the plane that passes through the points 
(1, 1, 1), (1, -1, 1), and (0, 0, 2).

Solution: Denote \( P = (1, 1, 1), \ Q = (1, -1, 1), \) and \( R = (0, 0, 2) \).
Then,
\[ \overrightarrow{PQ} = \langle 0, -2, 0 \rangle, \quad \overrightarrow{PR} = \langle -1, -1, 1 \rangle, \]

\[ \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{vmatrix} = (-2 - 0)i - (0 - 0)j + (0 - 2)k, \]

that is, \( \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2, 0, -2 \rangle. \) Take \( \mathbf{n} = \langle 2, 0, 2 \rangle. \)
With \( \mathbf{n} = \langle 2, 0, 2 \rangle \) and a point \( R = (0, 0, 2) \), the equation of the plane is
\[ 2(x - 0) + 0(y - 0) + 2(z - 2) = 0 \quad \Rightarrow \quad x + z = 2. \]
Example
Find the equation of the plane that is parallel to the plane 
\( x - 2y + 3z = 1 \) and passes through the center of the sphere
\( x^2 + 2x + y^2 + z^2 - 2z = 0 \).

Solution: The plane is parallel to the plane \( x - 2y + 3z = 1 \), so
their normal vectors are parallel. We choose \( \mathbf{n} = \langle 1, -2, 3 \rangle \).
We need to find the center of the sphere. We complete squares:
\[
0 = x^2 + 2x + y^2 + z^2 - 2z \\
= (x^2 + 2x + 1) - 1 + y^2 + (z^2 - 2z + 1) - 1 = 0 \\
= (x + 1)^2 + y^2 + (z - 1)^2 - 2.
\]

Therefore, the center of the sphere is at \( P_0 = (-1, 0, 1) \).
The equation of the plane is
\[
(x + 1) - 2(y - 0) + 3(z - 1) = 0 \quad \Rightarrow \quad x - 2y + 3z = 2.
\]

Example
Find the angle between the planes \( 2x - 3y + 2z = 1 \) and
\( x + 2y + 2z = 5 \).

Solution: The angle between the planes is the angle between their
normal vectors.
The normal vectors are \( \mathbf{n} = \langle 2, -3, 2 \rangle \), \( \mathbf{N} = \langle 1, 2, 2 \rangle \).
The cosine of the angle \( \theta \) between these vectors is
\[
\cos(\theta) = \frac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}| |\mathbf{N}|}.
\]
Since \( \mathbf{n} \cdot \mathbf{N} = 2 - 6 + 4 = 0 \), we conclude that \( \mathbf{n} \perp \mathbf{N} \).
The angle \( \theta \) is \( \theta = \pi/2 \).
Example
Find the vector equation for the line of intersection of the planes 
\[ 2x - 3y + 2z = 1 \] and 
\[ x + 2y + 2z = 5. \]

Solution: We first find the vector tangent to the line. This is a vector \( \mathbf{v} \) that belongs to both planes. This means that \( \mathbf{v} \) is perpendicular to both normal vectors \( \mathbf{n} = \langle 2, -3, 2 \rangle \) and \( \mathbf{N} = \langle 1, 2, 2 \rangle \).

One such vector is
\[
\mathbf{v} = \mathbf{n} \times \mathbf{N} = \begin{vmatrix}
i & j & k \\
2 & -3 & 2 \\
1 & 2 & 2 \\
\end{vmatrix} = \langle -6 - 4, -(4 - 2), (4 + 3) \rangle.
\]

So, \( \mathbf{v} = \langle -10, -2, 7 \rangle \).

Example
Find the vector equation for the line of intersection of the planes 
\[ 2x - 3y + 2z = 1 \] and 
\[ x + 2y + 2z = 5. \]

Solution: Recall \( \mathbf{v} = \langle -10, -2, 7 \rangle \). Now we need a point in the intersection of the planes. From the first plane we compute \( z \) as follows: 
\[ 2z = 1 - 2x + 3y. \]
We introduce this equation for \( 2z \) into the second plane:
\[ x + 2y + (1 - 2x + 3y) = 5 \quad \Rightarrow \quad -x + 5y = 4. \]

We need just one solution, so we choose: \( y = 0 \), then \( x = -4 \), and this implies \( z = 9/2 \). A point in the intersection of the planes is \( P_0 = (-4, 0, 9/2) \). The vector equation of the line is:
\[
\mathbf{r}(t) = \langle -4, -0, 9/2 \rangle + \langle -10, -2, 7 \rangle t.
\]
Example
Sketch the surface $36x^2 + 4y^2 + 9z^2 = 36$.

Solution: We first rewrite the equation above in the standard form

$$x^2 + \frac{4}{36} y^2 + \frac{9}{36} z^2 = 1 \iff x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1.$$ 

This is the equation of an ellipsoid with principal radius of length 1, 3, and 2 on the $x$, $y$ and $z$ axis, respectively. Therefore

Vector functions (Sect. 13.1).

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Integrals of vector functions.
Motion in space motivates to define vector functions.

**Definition**
A function \( r : I \to \mathbb{R}^n \), with \( n = 2, 3 \), is called a **vector function**, where the interval \( I \subset \mathbb{R} \) is called the **domain** of the function.

**Remark**: Given Cartesian coordinates in \( \mathbb{R}^3 \), the values of a vector function can be written in components as follows:

\[
r(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in I,
\]

where \( x(t), y(t), \) and \( z(t) \) are the values of three scalar functions.

Motion in space motivates to define vector functions.

**Remarks:**
- There is a natural association between a curve in \( \mathbb{R}^n \) and the vector function values \( r(t) \).
- The curve is determined by the terminal points of the vector function values \( r(t) \).
- The independent variable \( t \) is called the parameter of the curve.
**Example**

Graph the vector function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

**Solution:**

The curve given by \( \mathbf{r}(t) \) lies on a vertical cylinder with radius one, since

\[
x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1.
\]

The \( z(t) \) coordinate of the curve increases with \( t \), so the terminal point \( \mathbf{r}(t) \) moves up on the cylinder surface when \( t \) increases.

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**Example**

Graph the vector function \( \mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle \).

**Solution:**

The curve given by \( \mathbf{r}(t) \) lies on a horizontal cylinder with radius one, since

\[
x^2 + z^2 = \sin^2(t) + \cos^2(t) = 1.
\]

The \( y(t) \) coordinate of the curve increases with \( t \), so the terminal point \( \mathbf{r}(t) \) moves to the right on the cylinder surface when \( t \) increases.

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Vector functions (Sect. 13.1).

- Definition of vector functions: $r : \mathbb{R} \rightarrow \mathbb{R}^3$.
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Integrals of vector functions.

Limits and continuity of vector functions.

**Definition**
The vector function $r : I \rightarrow \mathbb{R}^n$, with $n = 2, 3$, has a *limit* given by the vector $L$ when $t$ approaches $t_0$, denoted as $\lim_{t \to t_0} r(t) = L$, iff the following holds: For every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|t - t_0| < \delta \quad \Rightarrow \quad |r(t) - L| < \epsilon.$$  

**Remark:**
- The limit of $r(t) = \langle x(t), y(t), z(t) \rangle$ as $t \to t_0$ is the limit of its components $x(t), y(t), z(t)$ in Cartesian coordinates.
- That is:
  $$\lim_{t \to t_0} r(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle.$$
\[ \lim_{t \to t_0} r(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle. \]

**Example**

Given \( r(t) = \langle \cos(t), \sin(t)/t, t^2 + 2 \rangle \), compute \( \lim_{t \to 0} r(t) \).

**Solution:**

Notice that the vector function \( r \) is not defined at \( t = 0 \), however its limit at \( t = 0 \) exists. Indeed,

\[
\lim_{t \to 0} r(t) = \lim_{t \to 0} \left\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2 \right\rangle
= \left\langle \lim_{t \to 0} \cos(t), \lim_{t \to 0} \frac{\sin(t)}{t}, \lim_{t \to 0} (t^2 + 2) \right\rangle
= \langle 1, 1, 2 \rangle.
\]

We conclude that \( \lim_{t \to 0} r(t) = \langle 1, 1, 2 \rangle \).  

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**Limits and continuity of vector functions.**

**Definition**

A vector function \( r : I \to \mathbb{R}^n \), with \( n = 2, 3 \), is **continuous at** \( t = t_0 \in I \) iff holds \( \lim_{t \to t_0} r(t) = r(t_0) \). The function \( r : I \to \mathbb{R}^n \) is **continuous** if it is continuous at every \( t \) in its domain interval \( I \).

**Remark:** A vector function with Cartesian components \( r = \langle x, y, z \rangle \) is continuous iff each component is continuous.

**Example**

The function \( r(t) = \langle \sin(t), t, \cos(t) \rangle \) is continuous for \( t \in \mathbb{R} \).  

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**Remark:** Having the idea of limit, one can introduce the idea of a derivative of a vector valued function.
Derivatives and motion.

Definition
The vector function \( r : I \to \mathbb{R}^n \), with \( n = 2, 3 \), is differentiable at \( t = t_0 \), denoted as \( r'(t) \) or \( \frac{dr}{dt} \), iff the following limit exists,

\[
r'(t) = \lim_{h \to 0} \frac{r(t + h) - r(t)}{h}.
\]

Remarks:
- A vector function \( r : I \to \mathbb{R}^n \) is differentiable if it is differentiable for each \( t \in I \).
- If a vector function with Cartesian components \( r = \langle x, y, z \rangle \) is differentiable, then

\[
r'(t) = \langle x'(t), y'(t), z'(t) \rangle.
\]
Derivatives and motion.

Theorem
If a vector function with Cartesian components \( \mathbf{r} = \langle x, y, z \rangle \) is differentiable, then \( \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \).

Proof.
\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h},
\]
\[
= \lim_{h \to 0} \langle \frac{x(t + h) - x(t)}{h}, \frac{y(t + h) - y(t)}{h}, \frac{z(t + h) - z(t)}{h} \rangle
\]
\[
= \langle \lim_{h \to 0} \frac{x(t + h) - x(t)}{h}, \lim_{h \to 0} \frac{y(t + h) - y(t)}{h}, \lim_{h \to 0} \frac{z(t + h) - z(t)}{h} \rangle
\]
\[
= \langle x'(t), y'(t), z'(t) \rangle.
\]

Example
Find the derivative of the vector function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle \).

Solution: We differentiate each component of \( \mathbf{r} \), that is,
\[
\mathbf{r}'(t) = \langle -\sin(t), \cos(t), (2t + 3) \rangle.
\]

Example
Find the derivative of the vector function \( \mathbf{r}(t) = \langle \cos(2t), e^{3t}, 1/t \rangle \).

Solution: We differentiate each component of \( \mathbf{r} \), that is,
\[
\mathbf{r}'(t) = \langle -2\sin(2t), 3e^{3t}, -1/t^2 \rangle.
\]
Geometrical property of the derivative.

Remark: The vector $r'(t)$ is tangent to the curve given by $r$ at the point $r(t)$.

Remark: If $r(t)$ represents the vector position of a particle, then:
- The derivative of the position function is the velocity function, $v(t) = r'(t)$. The speed is $|v(t)|$.
- The derivative of the velocity function is the acceleration function, $a(t) = v'(t) = r''(t)$.

Derivatives and motion.

Example
Compute the derivative of the position function $r(t) = \langle \cos(t), \sin(t), 0 \rangle$. Graph the curve given by $r$, and explicitly show the position vector $r(0)$ and velocity vector $v(0)$.

Solution:
The derivative of $r$ is:

$$v(t) = \langle -\sin(t), \cos(t), 0 \rangle.$$ 

$r(0) = \langle 1, 0, 0 \rangle$, $v(0) = \langle 0, 1, 0 \rangle$. 
Differentiation rules are the same as for scalar functions

**Theorem**
If \( \mathbf{v} \) and \( \mathbf{w} \) are differentiable vector functions, then holds:

- \( [\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t) \), \quad \text{(addition)};
- \( [c\mathbf{v}(t)]' = c\mathbf{v}'(t) \), \quad \text{(product rule)};
- \( [\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t) \), \quad \text{(chain rule)};
- \( [f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t) \), \quad \text{(product rule)};
- \( [\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t) \), \quad \text{(dot product)};
- \( [\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t) \), \quad \text{(cross product)}.

Higher derivatives can also be computed.

**Remark:** The \( m \)-derivative of a vector function \( \mathbf{r} \) is denoted as \( \mathbf{r}^{(m)} \) and is given by the expression \( \mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]' \).

**Example**
Compute the third derivative of \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle \).

**Solution:**

\[
\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\
\mathbf{r}^{(2)}(t) = (\mathbf{r}'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle, \\
\mathbf{r}^{(3)}(t) = (\mathbf{r}^{(2)}(t))' = \langle \sin(t), -\cos(t), 0 \rangle.
\]

**Recall:** If \( \mathbf{r}(t) \) is the position of a particle, then \( \mathbf{v}(t) = \mathbf{r}'(t) \) is the velocity and \( \mathbf{a}(t) = \mathbf{r}^{(2)}(t) \) is the acceleration of the particle.
Vector functions (Sect. 13.1).

- Definition of vector functions: \( r : \mathbb{R} \to \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- **Integrals of vector functions.**

Integrals of vector functions.

**Definition**

The *indefinite integral*, also called the *antiderivative*, of a vector function \( \mathbf{v} \) is denoted as \( \int \mathbf{v}(t) \, dt \) and given by

\[
\int \mathbf{v}(t) \, dt = \mathbf{V}(t) + \mathbf{C},
\]

where \( \mathbf{V}'(t) = \mathbf{v}(t) \) and \( \mathbf{C} \) is a constant vector.

**Example**

Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

**Solution:** The position function is the primitive of the velocity function, \( \mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C} \), that satisfies the initial condition \( \mathbf{r}(0) = \mathbf{V}(0) + \mathbf{C} \). This initial condition fixes the constant vector \( \mathbf{C} \).
Integrals of vector functions.

Example
Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

Solution: The position function is a primitive of the velocity,
\[
\mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,
\]
with \( \mathbf{C} = \langle c_x, c_y, c_z \rangle \) a constant vector. The initial condition determines the vector \( \mathbf{C} \):
\[
\langle 1, 1, 1 \rangle = \mathbf{r}(0) = \mathbf{V}(0) + \mathbf{C} = \langle 0, 0, -1 \rangle + \langle c_x, c_y, c_z \rangle,
\]
that is, \( c_x = 1, c_y = 1, c_z = 2 \).

The position function is \( \mathbf{r}(t) = \langle t^2 + 1, \sin(t) + 1, -\cos(t) + 2 \rangle \).

\[\boxed{\text{ Integrals of vector functions. }}\]

Example
Find the position function of a particle with acceleration \( \mathbf{a}(t) = \langle 0, 0, -10 \rangle \) having an initial velocity \( \mathbf{v}(0) = \langle 0, 1, 1 \rangle \) and initial position \( \mathbf{r}(0) = \langle 1, 0, 1 \rangle \).

Solution: The velocity is \( \mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle \).

The initial condition implies \( \langle 0, 1, 1 \rangle = \mathbf{v}(0) = \langle v_{0x}, v_{0y}, v_{0z} \rangle \), that is \( v_{0x} = 0, v_{0y} = 1, v_{0z} = 1 \). The velocity function is
\[
\mathbf{v}(t) = \langle 0, 1, (-10t + 1) \rangle.
\]

The position is \( \mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z}) \rangle \).

The initial condition implies \( \langle 1, 0, 1 \rangle = \mathbf{r}(0) = \langle r_{0x}, r_{0y}, r_{0z} \rangle \), that is \( r_{0x} = 1, r_{0y} = 0, r_{0z} = 1 \). The velocity function is
\[
\mathbf{r}(t) = \langle 1, t, (-5t^2 + t + 1) \rangle.
\]

\[\boxed{\text{ Integrals of vector functions. }}\]
Integrals of vector functions.

**Definition**
If the components of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ are integrable functions on the interval $[a, b]$, then the *definite integral* of $\mathbf{r}$ is given by

$$
\int_a^b \mathbf{r}(t) \, dt = \left\langle \int_a^b x(t) \, dt, \int_a^b y(t) \, dt, \int_a^b z(t) \, dt \right\rangle.
$$

**Example**
Compute $\int_0^\pi \mathbf{r}(t) \, dt$ for the function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

**Solution:**

$$
\int_0^\pi \mathbf{r}(t) \, dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle \, dt
$$

$$
= \left\langle \int_0^\pi \cos(t) \, dt, \int_0^\pi \sin(t) \, dt, \int_0^\pi t \, dt \right\rangle,
$$

$$
= \left\langle \sin(t) \bigg|_0^\pi, -\cos(t) \bigg|_0^\pi, \frac{t^2}{2} \bigg|_0^\pi \right\rangle,
$$

$$
= \left\langle 0, 2, \frac{\pi^2}{2} \right\rangle, \quad \Rightarrow \quad \int_0^\pi \mathbf{r}(t) \, dt = \left\langle 0, 2, \frac{\pi^2}{2} \right\rangle.
$$

$\triangle$
The arc length of a curve in space (Sect. 13.3).

- The arc length of a curve in space.
- The arc length function.
- Parametrizations of a curve.
- The arc length parametrization of a curve.

The length of a curve is called its arc length.

Definition
The arc length of a continuously differentiable curve \( \mathbf{r} : [a, b] \rightarrow \mathbb{R}^n \), with \( n=2,3 \), is the number given by

\[
\ell_{ba} = \int_a^b |\mathbf{r}'(t)| \, dt.
\]

Remark:
- If the curve \( \mathbf{r} \) is the path traveled by a particle in space, then \( \mathbf{r}' = \mathbf{v} \) is the velocity of the particle.
- The arc length is the integral in time of the particle speed \( |\mathbf{v}(t)| \).
- Therefore, the arc length of the curve is the distance traveled by the particle.
The length of a curve is called its arc length.

Recall:
The arc length of a curve \( r : [a, b] \to \mathbb{R}^3 \)

\[
\ell_{ba} = \int_a^b |r'(t)| \, dt.
\]

Remark:
In Cartesian coordinates the functions \( r \) and \( r' \) are given by

\[
r(t) = \langle x(t), y(t), z(t) \rangle, \quad r'(t) = \langle x'(t), y'(t), z'(t) \rangle.
\]

Therefore the arc length of the curve is given by the expression

\[
\ell_{ba} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.
\]

The arc length of a curve in a plane.

Example
Find the arc length of the curve \( r(t) = \langle \cos(t), \sin(t) \rangle \), for \( t \in [\pi/4, 3\pi/4] \).

Solution: The derivative vector function is

\( r'(t) = \langle -\sin(t), \cos(t) \rangle \). The arc length formula is

\[
l = \int_{\pi/4}^{3\pi/4} \sqrt{[-\sin(t)]^2 + [\cos(t)]^2} \, dt
\]

\[
= \int_{\pi/4}^{3\pi/4} \, dt \quad \Rightarrow \quad l = \frac{\pi}{2}.
\]

This result is reasonable, since the curve is a circle and we are computing the length of quarter a circle.
The arc length of a curve in a plane.

Example
Find the arc length of the spiral \( r(t) = \langle t \cos(t), t \sin(t) \rangle \), for \( t \in [0, t_0] \).

Solution: The derivative vector is
\[
r'(t) = \langle -t \sin(t) + \cos(t), t \cos(t) + \sin(t) \rangle.
\]

\[
|r'(t)|^2 = \left[ t^2 \sin^2(t) + \cos^2(t) - 2t \sin(t) \cos(t) \right] ^2 + \left[ t^2 \cos^2(t) + \sin^2(t) + 2t \sin(t) \cos(t) \right] = t^2 + 1.
\]

The arc length is \( \ell(t_0) = \int_0^{t_0} \sqrt{1 + t^2} \, dt = \ln(t + \sqrt{1 + t^2}) \bigg|_0^{t_0} \).

We conclude: \( \ell(t_0) = \ln(t_0 + \sqrt{1 + t_0^2}). \) \( \triangleright \)

The arc length of a curve in space.

Example
Find the arc length of
\( r(t) = \langle 6 \cos(2t), 6 \sin(2t), 5t \rangle \), for \( t \in [0, \pi] \).

Solution: The derivative vector is
\[
r'(t) = \langle -12 \sin(2t), 12 \cos(2t), 5 \rangle,
\]
\[
|r'(t)|^2 = 144 \left[ \sin^2(2t) + \cos^2(2t) \right] + 25 = 169 = (13)^2.
\]

The arc length is \( \ell = \int_0^\pi 13 \, dt = 13 \pi \bigg|_0^\pi \Rightarrow \ell = 13 \pi. \) \( \triangleright \)
Idea behind the arc length formula.

The arc length formula can be obtained as a limit procedure. One adds up the lengths of a polygonal line that approximates the original curve.

\[
\ell_N = \sum_{n=0}^{N-1} |\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)|, \quad \{a = t_0, t_1, \cdots, t_{N-1}, t_N = b\},
\]

\[
\simeq \sum_{n=0}^{N-1} |\mathbf{r}'(t_n)| (t_{n+1} - t_n) \xrightarrow{N \to \infty} \int_a^b |\mathbf{r}'(t)| \, dt
\]

The arc length of a curve in space (Sect. 13.3).

- The arc length of a curve in space.
- **The arc length function.**
- Parametrizations of a curve.
- The arc length parametrization of a curve.
The arc length function.

Definition
The arc length function of a continuously differentiable vector function \( \mathbf{r} \) is given by

\[
\ell(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| \, d\tau.
\]

Remarks:
- The value \( \ell(t) \) of the arc length function represents the length along the curve \( \mathbf{r} \) from \( t_0 \) to \( t \).
- If the function \( \mathbf{r} \) is the position of a moving particle as function of time, then the arc length \( \ell(t) \) is the distance traveled by the particle from the time \( t_0 \) to \( t \).

Example
Find the arc length function for the curve \( \mathbf{r}(t) = \langle 6 \cos(2t), 6 \sin(2t), 5t \rangle \), starting at \( t = 0 \).

Solution: We have found that \( |\mathbf{r}'(t)| = 13 \). Therefore,

\[
\ell(t) = \int_{0}^{t} 13 \, d\tau \quad \Rightarrow \quad \ell(t) = 13t.
\]
The arc length function.

Example
Given the position function in time \( r(t) = \langle 6 \cos(2t), 6 \sin(2t), 5t \rangle \), find the position vector \( r(t_0) \) located at a length \( \ell_0 = 20 \) from the initial position \( r(0) \).

Solution: We have found that the arc length function for the vector function \( r \) is \( \ell(t) = 13t \).
Since \( t = \ell/13 \), the time at \( \ell = \ell_0 = 20 \) is \( t_0 = 13/20 \).
Therefore, the position vector at \( \ell_0 = 20 \) is given by
\[
r(t_0) = \langle 6 \cos(13/10), 6 \sin(13/10), 13/4 \rangle.
\]

The arc length of a curve in space (Sect. 13.3).

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Parametrizations of a curve.

Remark:
A curve in space can be represented by different vector functions.

Example
The unit circle in $\mathbb{R}^2$ is the curve represented by the following vector functions:

$\begin{align*}
\mathbf{r}_1(t) &= \langle \cos(t), \sin(t) \rangle; \\
\mathbf{r}_2(t) &= \langle \cos(5t), \sin(5t) \rangle; \\
\mathbf{r}_3(t) &= \langle \cos(e^t), \sin(e^t) \rangle.
\end{align*}$

Remark:
The curve in space is the same for all three functions above. The vector $\mathbf{r}$ moves along the curve at different speeds for the different parametrizations.

Parametrizations of a curve.

Remarks:
$\begin{align*}
\item If the vector function $\mathbf{r}$ represents the position in space of a moving particle, then there is a preferred parameter to describe the motion: The time $t$.
\item Another parameter that is useful to describe a moving particle is the distance traveled by the particle, the arc length $\ell$.
\item A common problem is the following: Given a vector function parametrized by the time $t$, switch the curve parameter to the arc length $\ell$.
\item The problem above is called the arc length parametrization of a curve.
\end{align*}$
The arc length of a curve in space (Sect. 13.3).

- The arc length of a curve in space.
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- **The arc length parametrization of a curve.**

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The arc length parametrization of a curve.

**Problem:**
Given vector function \( \mathbf{r} \) in terms of a parameter \( t \), find the arc length parametrization of that curve.

**Solution:**
- With the function values \( \mathbf{r}(t) \) compute the arc length function \( \ell(t) \), starting at some \( t = t_0 \).
- Invert the function values \( \ell(t) \) to find the function values \( t(\ell) \).
- Example: If \( \ell(t) = 3e^{t/2} \), then \( t(\ell) = 2\ln(\ell/3) \).
- Compute the composition function \( \mathbf{r}(\ell) = \mathbf{r}(t(\ell)) \). That is, replace \( t \) by \( t(\ell) \) in the function values \( \mathbf{r}(t) \).

The function values \( \mathbf{r}(\ell) \) are the parametrization of the function values \( \mathbf{r}(t) \) using the arc length as the new parameter.
The arc length parametrization of a curve.

**Example**
Find the arc length parametrization of the vector function \( \mathbf{r}(t) = \langle 4 \cos(t), 4 \sin(t), 3t \rangle \) starting at \( t = 0 \).

**Solution:** First find the derivative function:
\[
\mathbf{r}'(t) = \langle -4 \sin(t), 4 \cos(t), 3 \rangle.
\]
Hence, \( |\mathbf{r}'(t)|^2 = 4^2 \sin^2(t) + 4^2 \cos^2(t) + 3^2 = 16 + 9 = 5^2 \).
Find the arc length function: \( \ell(t) = \int_0^t 5 \, d\tau \Rightarrow \ell(t) = 5t \).
Invert the equation above: \( t = \ell/5 \).
Reparametrize the original curve:
\[
\mathbf{r}(\ell) = \langle 4 \cos(\ell/5), 4 \sin(\ell/5), 3\ell/5 \rangle.
\]

The arc length parametrization of a curve.

**Theorem**
A unit tangent vector to a curve given by the vector function values \( \mathbf{r}(t) \) is given by \( \mathbf{u}(\ell) = \frac{d\mathbf{r}}{d\ell} \), where \( \ell \) is the arc length of the curve.

**Proof.**
Given the function values \( \mathbf{r}(t) \), let \( \mathbf{r}(\ell) \) be the reparametrization of \( \mathbf{r}(t) \) with the arc length function \( \ell(t) = \int_{t_0}^t |\mathbf{r}'(\tau)| \, d\tau \).
Notice that \( \frac{d\ell}{dt} = |\mathbf{r}'(t)| \) and \( \frac{dt}{d\ell} = \frac{1}{|\mathbf{r}'(t)|} \).
Therefore, \( \mathbf{u}(\ell) = \frac{d\mathbf{r}(\ell)}{d\ell} = \frac{d\mathbf{r}(t)}{dt} \frac{dt}{d\ell} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \).
We conclude that \( |\mathbf{u}(\ell)| = 1 \). \( \square \)
Example

Find a unit vector tangent to the curve given by
\[ \mathbf{r}(t) = \langle 4 \cos(t), 4 \sin(t), 3t \rangle \text{ for } t \geq 0. \]

Solution: Reparametrize the curve using the arc length. We get
\[ \mathbf{r}(\ell) = \langle 4 \cos(\ell/5), 4 \sin(\ell/5), 3\ell/5 \rangle. \]

Therefore, a unit tangent vector is
\[ \mathbf{u}(\ell) = \frac{d\mathbf{r}}{d\ell} \Rightarrow \mathbf{u}(\ell) = \langle -\frac{4}{5} \sin(\ell/5), \frac{4}{5} \cos(\ell/5), \frac{3}{5} \rangle. \]

We can verify that this is a unit vector, since
\[ |\mathbf{u}(\ell)|^2 = \left(\frac{4}{5}\right)^2 \left[ \sin^2(\ell/5) + \cos^2(\ell/5) \right] + \left(\frac{3}{5}\right)^2 \Rightarrow |\mathbf{u}(\ell)| = 1. \]