Lines and planes in space (Sect. 12.5)

Lines in space (Today).

- Review: Lines on a plane.
- The equations of lines in space:
  - Vector equation.
  - Parametric equation.
- Distance from a point to a line.

Planes in space (Next class).

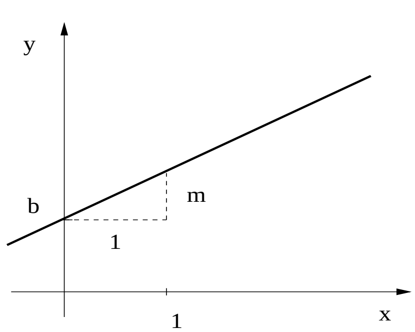
- Equations of planes in space.
  - Vector equation.
  - Components equation.
- The line of intersection of two planes.
- Parallel planes and angle between planes.
- Distance from a point to a plane.

Review: Lines on a plane

Equation of a line

The equation of a line with slope m and vertical intercept b is given by

$$y = mx + b.$$

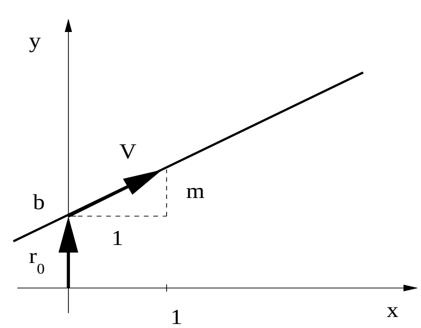


Vector equation of a line

The equation of the line by the point $P = (0, b)$ parallel to the vector $\mathbf{v} = \langle 1, m \rangle$ is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v},$$

where $\mathbf{r}_0 = \overrightarrow{OP} = \langle 0, b \rangle$.



Review: Lines on a plane

Example

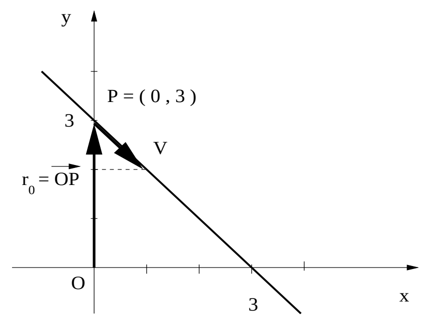
Find the vector equation of a line $y = -x + 3$.

Solution: The vertical intercept is at the point $P = (0, 3)$.

A vector tangent to the line is $\mathbf{v} = \langle 1, -1 \rangle$, since the point $P_1 = (1, 2)$ belongs to the line, which implies that $\mathbf{v} = \overrightarrow{PP_1} = \langle (1 - 0), (2 - 3) \rangle = \langle 1, -1 \rangle$.

The vector equation for the line is

$$\mathbf{r}(t) = \langle 0, 3 \rangle + t \, \langle 1, -1 \rangle.$$



$\triangleleft$

Review: Lines on a plane

We verify the result above: That the line $y = -x + 3$ is indeed

$$\mathbf{r}(t) = \langle 0, 3 \rangle + t \, \langle 1, -1 \rangle, \quad \text{(Vector equation of the line.)}$$

If $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then $\langle x(t), y(t) \rangle = \langle (0 + t), (3 - t) \rangle$. That is,

$$x(t) = t, \qquad \text{(Parametric equation of the line.)}$$
$$y(t) = 3 - t. \qquad \text{(The parameter is } t.\text{)}$$

Replacing t by x is the second equation above we obtain

$$y(x) = -x + 3.$$

Review: Lines on a plane

Vector equation of a line

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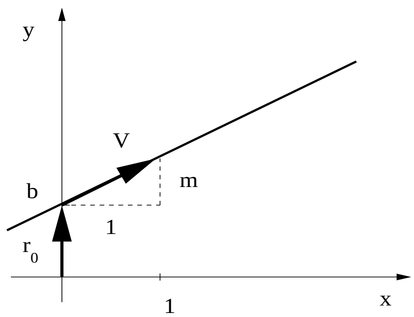

Parametric equation of a line

A line with vector equation

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v},$$

where $\mathbf{r}_0 = \langle 0, b \rangle$ and $\mathbf{v} = \langle 1, m \rangle$ can also be written as follows

$$\langle x(t), y(t) \rangle = \langle (0 + t), (b + tm) \rangle,$$

that is,

$$x(t) = t$$
$$y(t) = b + mt.$$

Lines and planes in space (Sect. 12.5)

Lines in space

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A line is specified by a point and a tangent vector

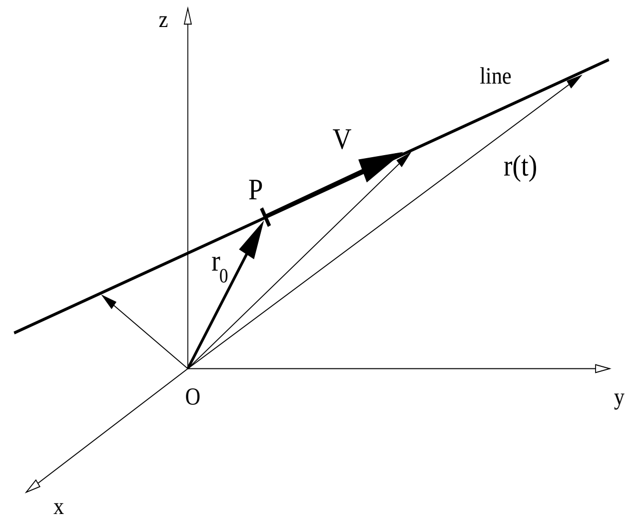
Vector equation of a line

Definition

Fix Cartesian coordinates in $\mathbb{R}^3$ with origin at a point O. Given a point P and a vector $\mathbf{v}$ in $\mathbb{R}^3$, the *line by P parallel to $\mathbf{v}$* is the set of terminal points of the vectors

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}, \quad t \in \mathbb{R},$$

where $\mathbf{r}_0 = \overrightarrow{OP}$.



We refer to a line to mean both the set of vectors $\mathbf{r}(t)$ and the set of terminal points of these vectors.

Vector equation of a line.

Example

Find the vector equation of the line by the point $P = (1, -2, 1)$ tangent to the vector $\mathbf{v} = \langle 1, 2, 3 \rangle$.

Solution:

The vector $\mathbf{r}_0 = \overrightarrow{OP} = \langle 1, -2, 1 \rangle$, therefore, the formula $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$ implies

$$\mathbf{r}(t) = \langle 1, -2, 1 \rangle + t\,\langle 1, 2, 3 \rangle.$$

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Lines and planes in space (Sect. 12.5)

Lines in space

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Parametric equation of a line.

Definition

The *parametric equations of a line* by $P = (x_0, y_0, z_0)$ tangent to $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ are given by

$$\begin{aligned} x(t) &= x_0 + t\, v_x, \\ y(t) &= y_0 + t\, v_y, \\ z(t) &= z_0 + t\, v_z. \end{aligned}$$

Remark: It is simple to obtain the parametric equations form the vector equation, and vice-versa, noticing the relation

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + t\,\mathbf{v} \\ \langle x(t), y(t), z(t) \rangle &= \langle x_0, y_0, z_0 \rangle + t\,\langle v_x, v_y, v_z \rangle \\ &= \langle (x_0 + t\,v_x), (y_0 + t\,v_y), (z_0 + t\,v_z) \rangle. \end{aligned}$$

Parametric equation of a line.

Example

Find the parametric equations of the line with vector equation

$$\mathbf{r}(t) = \langle 1, -2, 1\rangle + t\,\langle 1, 2, 3\rangle.$$

Solution: Rewrite the vector equation in vector components,

$$\langle x(t), y(t), z(t)\rangle = \langle (1+t), (-2+2t), (1+3t)\rangle.$$

We conclude that

$$x(t) = 1 + t,$$
$$y(t) = -2 + 2t,$$
$$z(t) = 1 + 3t.$$

$\lhd$

Parametric equation of a line.

Example

Find both the vector equation and the parametric equation of the line containing the points $P = (1, 2, -3)$ and $Q = (3, -2, 1)$.

Solution: A vector tangent to the line is $\mathbf{v} = \overrightarrow{PQ}$, which is given by

$$\mathbf{v} = \langle (3-1), (-2-2), (1+3)\rangle \quad \Rightarrow \quad \mathbf{v} = \langle 2, -4, 4\rangle.$$

We can use either P or Q to express the vector equation for the line. If we use P, then the vector equation of the line is

$$\mathbf{r}(t) = \langle 1, 2, -3\rangle + t\,\langle 2, -4, 4\rangle.$$

If we choose Q, the vector equation of the line is

$$\mathbf{r}(s) = \langle 3, -2, 1\rangle + s\,\langle 2, -4, 4\rangle.$$

We use s to do not confuse it with the t above .

Parametric equation of a line.

Example

Find both the vector equation and the parametric equation of the line containing the points $P = (1, 2, -3)$ and $Q = (3, -2, 1)$.

Solution: The parametric equation of the line is simple to obtain once the vector equation is known. Since

$$\mathbf{r}(t) = \langle 1, 2, -3\rangle + t\,\langle 2, -4, 4\rangle,$$

then $\langle x(t), y(t), z(t)\rangle = \langle (1 + 2t), (2 - 4t), (-3 + 4t)\rangle$.
Then, the parametric equations of the line are given by

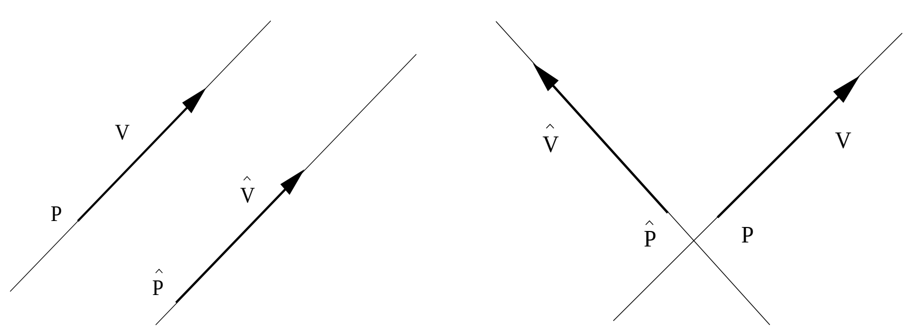
$$x(t) = 1 + 2t,$$
$$y(t) = 2 - 4t,$$
$$z(t) = -3 + 4t.$$

$\triangleleft$

Parallel lines, perpendicular lines, intersections

Definition

The lines $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$ and $\hat{\mathbf{r}}(t) = \hat{\mathbf{r}}_0 + t\,\hat{\mathbf{v}}$ are *parallel* iff their tangent vectors $\mathbf{v}$ and $\hat{\mathbf{v}}$ are parallel; they are *perpendicular* iff $\mathbf{v}$ and $\hat{\mathbf{v}}$ are perpendicular; and the lines *intersect* iff they have a common point.

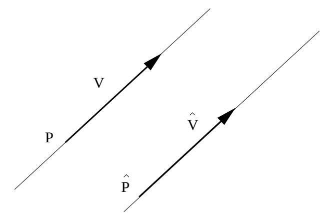


Perpendicular lines in space may not intersect.
Non-parallel lines in space may not intersect.

Parallel lines, perpendicular lines, intersections

Example

Find the line through $P = (1,1,1)$ and parallel to the line $\hat{\mathbf{r}}(t) = \langle 1,2,3\rangle + t\,\langle 2,-1,1\rangle$



Solution:

We need to find $\mathbf{r}_0$ and $\mathbf{v}$ such that $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$.

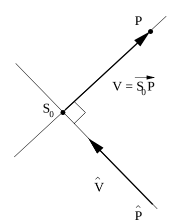
The vector $\mathbf{r}_0$ is simple to find: $\mathbf{r}_0 = \overrightarrow{OP} = \langle 1,1,1\rangle$.

The vector $\mathbf{v}$ is simple to find too: $\mathbf{v} = \langle 2,-1,1\rangle$.

We conclude: $\mathbf{r}(t) = \langle 1,1,1\rangle + t\,\langle 2,-1,1\rangle$. $\triangleleft$

Example

Find the line through $P = (1,1,1)$ perpendicular to and intersecting the line $\hat{\mathbf{r}}(t) = \langle 1,2,3\rangle + t\,\langle 2,-1,1\rangle$



Solution:

Find a point S on the intersection such that $\overrightarrow{PS}$ is perpendicular to $\hat{\mathbf{v}} = \langle 2,-1,1\rangle$. Writing $S_t = \hat{\mathbf{r}}(t) = \langle (1+2t),(2-t),(3+t)\rangle$,

$$\overrightarrow{PS_t} = \langle 2t,(1-t),(2+t)\rangle \perp \hat{\mathbf{v}} = \langle 2,-1,1\rangle \quad\Leftrightarrow\quad \overrightarrow{PS_t}\cdot\hat{\mathbf{v}} = 0.$$

$$0 = \overrightarrow{PS_t}\cdot\hat{\mathbf{v}} = 4t + (-1+t) + (2+t) = 6t+1 \quad\Rightarrow\quad t_0 = -\frac{1}{6}.$$

$$\overrightarrow{PS_0} = \Big\langle -\frac{2}{6}, \Big(1+\frac{1}{6}\Big), \Big(2-\frac{1}{6}\Big)\Big\rangle \quad\Rightarrow\quad \overrightarrow{PS_0} = \frac{1}{6}\langle -2,7,11\rangle.$$

$$\mathbf{r}(t) = \overrightarrow{OP} + t\,\overrightarrow{PS_0} \quad\Rightarrow\quad \mathbf{r}(t) = \langle 1,1,1\rangle + \frac{t}{6}\,\langle -2,7,11\rangle.$$

$\triangleleft$

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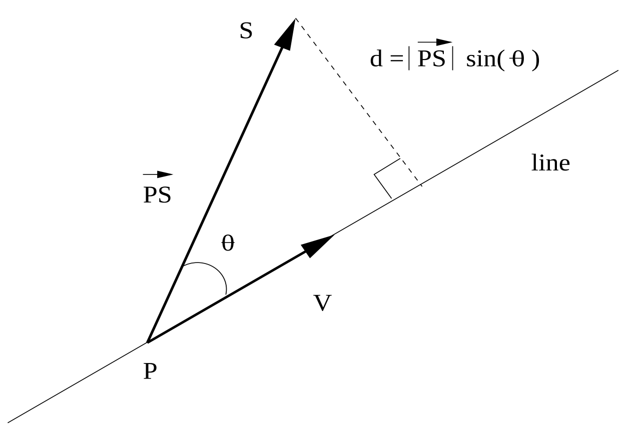
Distance from a point to a line.

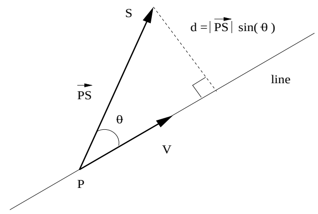
Theorem

The distance from a point S in space to a line through the point P with tangent vector $\mathbf{v}$ is given by

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}\,|}{|\mathbf{v}\,|}.$$



Distance from a point to a line.



Proof.

The distance from the point S to the line passing by the point P with tangent vector $\mathbf{v}$ is given by

$$d = |\overrightarrow{PS}| \sin(\theta).$$

Recalling that $|\overrightarrow{PS} \times \mathbf{v}| = |\overrightarrow{PS}|\,|\mathbf{v}| \sin(\theta)$, we conclude that

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

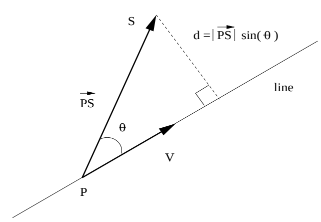
□

Distance from a point to a line.

Example

Find the distance from the point $S = (1, 2, 1)$ to the line

$$x = 2 - t, \quad y = -1 + 2t, \quad z = 2 + 2t.$$



Solution:

First we need to compute the vector equation of the line above.

This line has tangent vector $\mathbf{v} = \langle -1, 2, 2 \rangle$.

(The vector components are the numbers that multiply t.)

This line contains the vector $P = (2, -1, 2)$.

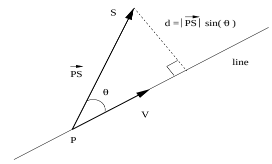
(Just evaluate the line above at $t = 0$.)

Therefore, $\overrightarrow{PS} = \langle -1, 3, -1 \rangle$.

Example

Find the distance from the point $S = (1, 2, 1)$ to the line

$$x = 2 - t, \quad y = -1 + 2t, \quad z = 2 + 2t.$$



Solution:

So far: $P = (2, -1, 2)$, $\mathbf{v} = \langle -1, 2, 2 \rangle$, and $\overrightarrow{PS} = \langle -1, 3, -1 \rangle$.
Since $d = |\overrightarrow{PS} \times \mathbf{v}|/|\mathbf{v}|$, we need to compute:

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -1 \\ -1 & 2 & 2 \end{vmatrix} = (6+2)\,\mathbf{i} - (-2-1)\,\mathbf{j} + (-2+3)\,\mathbf{k},$$

that is, $\overrightarrow{PS} \times \mathbf{v} = \langle 8, 3, 1 \rangle$. We then compute the lengths:

$$|\overrightarrow{PS} \times \mathbf{v}| = \sqrt{64 + 9 + 1} = \sqrt{74}, \quad |\mathbf{v}| = \sqrt{1 + 4 + 4} = 3.$$

The distance from S to the line is $d = \sqrt{74}/3$.

Exercise

Consider the lines

$$x(t) = 1 + t, \qquad x(s) = 2s,$$
$$y(t) = \frac{3}{2} + 3t, \qquad y(s) = 1 + s,$$
$$z(t) = -t, \qquad z(s) = -2 + 4s.$$

Are the lines parallel? Do they intersect?

Answer:

The lines are not parallel.

The lines intersect at $P = \left(1, \frac{3}{2}, 0\right)$. $\triangleleft$

Lines and planes in space (Sect. 12.5)

Planes in space.

- Equations of planes in space.
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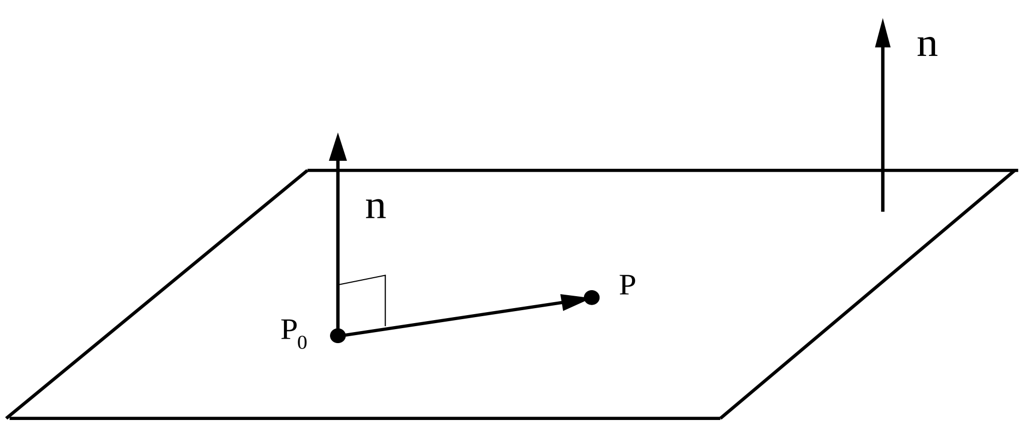
A point an a vector determine a plane.

Definition

Given a point P_0 and a non-zero vector $\mathbf{n}$ in $\mathbb{R}^3$, the *plane by P_0 perpendicular to* $\mathbf{n}$ is the set of points P solution of the equation

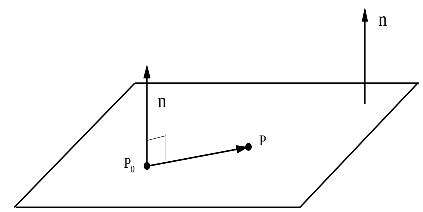
$$(\overrightarrow{P_0P}) \cdot \mathbf{n} = 0.$$



A point an a vector determine a plane.

Example

Does the point $P = (1, 2, 3)$ belong to the plane containing $P_0 = (3, 1, 2)$ and perpendicular to $\mathbf{n} = \langle 1, 1, 1\rangle$?



Solution: We need to know if the vector $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$. We first compute $\overrightarrow{P_0P}$ as follows,

$$\overrightarrow{P_0P} = \langle (1-3), (2-1), (3-2)\rangle \quad \Rightarrow \quad \overrightarrow{P_0P} = \langle -2, 1, 1\rangle.$$

This vector is orthogonal to $\mathbf{n}$, since

$$\big(\overrightarrow{P_0P}\big) \cdot \mathbf{n} = -2 + 1 + 1 = 0.$$

We conclude that P belongs to the plane. $\triangleleft$

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Equation of a plane in Cartesian coordinates

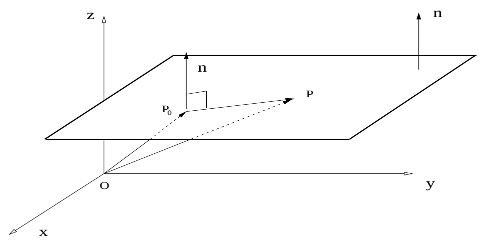
Theorem

Given any Cartesian coordinate system, the point $P = (x, y, z)$ belongs to the plane by $P_0 = (x_0, y_0, z_0)$ perpendicular to $\mathbf{n} = \langle n_x, n_y, n_z \rangle$ iff holds

$$(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0.$$

Furthermore, the equation above can be written as

$$n_x x + n_y y + n_z z = d, \qquad d = n_x x_0 + n_y y_0 + n_z z_0.$$



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Furthermore, the equation above can be written as

$$n_x x + n_y y + n_z z = d, \qquad d = n_x x_0 + n_y y_0 + n_z z_0.$$

Proof.

In Cartesian coordinates $\overrightarrow{P_0P} = \langle (x - x_0), (y - y_0), (z - z_0) \rangle$.
Therefore, the equation of the plane is

$$0 = \left(\overrightarrow{P_0P}\right) \cdot \mathbf{n} = (x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z.$$

$\square$

Equation of a plane in Cartesian coordinates

Example

Find the equation of a plane containing $P_0 = (1, 2, 3)$ and perpendicular to $\mathbf{n} = \langle 1, -1, 2 \rangle$.

Solution: The point $P = (x, y, z)$ belongs to the plane above iff $\big(\overrightarrow{P_0P}\big) \cdot \mathbf{n} = 0$, that is,

$$\langle (x-1), (y-2), (z-3) \rangle \cdot \langle 1, -1, 2 \rangle = 0.$$

Computing the dot product above we get

$$(x-1) - (y-2) + 2(z-3) = 0.$$

The equation of the plane can be also written as

$$x - y + 2z = 5.$$

$\triangleleft$

Equation of a plane in Cartesian coordinates

Example

Find a point P_0 and the perpendicular vector $\mathbf{n}$ to the plane $2x + 4y - z = 3$.

Solution: We know that the general equation of a plane is

$$n_x x + n_y y + n_z z = d.$$

The components of the vector $\mathbf{n}$, called *normal vector*, are the coefficients that multiply the variables x, y and z. Therefore,

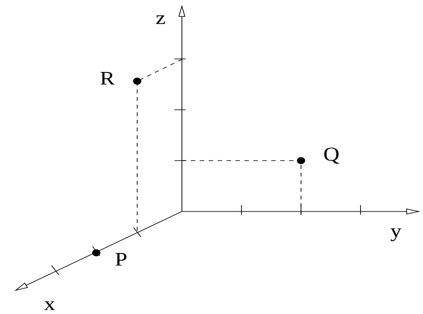
$$\mathbf{n} = \langle 2, 4, -1 \rangle.$$

A point P_0 on the plane is simple to find. Just look for the intersection of the plane with one of the coordinate axis.
For example: set $y = 0$, $z = 0$ and find x from the equation of the plane: $2x = 3$, that is $x = 3/2$. Therefore, $P_0 = (3/2, 0, 0)$. $\triangleleft$

Equation of a plane in Cartesian coordinates

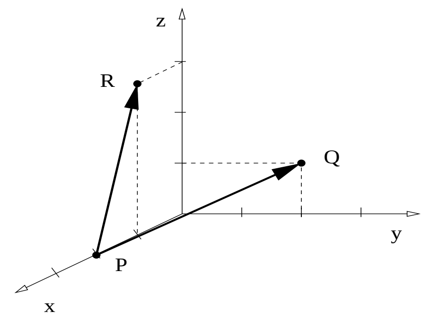
Example

Find the equation of the plane containing the points $P = (2, 0, 0)$, $Q = (0, 2, 1)$, $R = (1, 0, 3)$.



Solution:

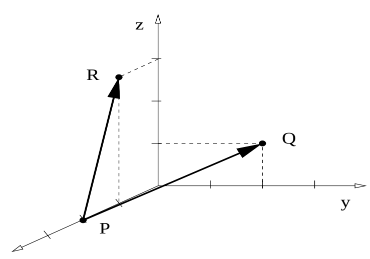
Find two tangent vectors to the plane, for example, $\overrightarrow{PQ} = \langle -2, 2, 1 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$.



Equation of a plane in Cartesian coordinates

Solution:

Find two tangent vectors to the plane, for example, $\overrightarrow{PQ} = \langle -2, 2, 1 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$.



A normal vector $\mathbf{n}$ to the plane tangent to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ can be obtained using the cross product: $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$. That is,

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ -1 & 0 & 3 \end{vmatrix} = (6 - 0)\,\mathbf{i} - (-6 + 1)\,\mathbf{j} + (0 + 2)\,\mathbf{k}.$$

The result is: $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 6, 5, 2 \rangle$.
Choose any point on the plane, say $P = (2, 0, 0)$.
Then, the equation of the plane is: $6(x - 2) + 5y + 2z = 0$. $\lhd$

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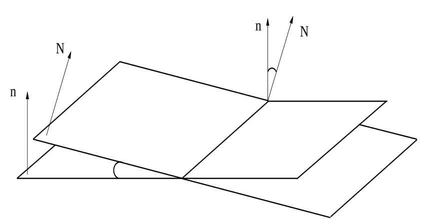
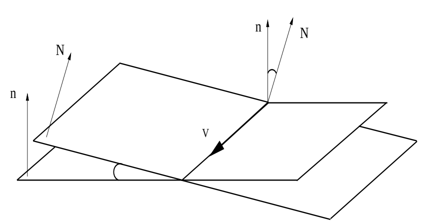
The line of intersection of two planes.





Example

Find a vector tangent to the line of intersection of the planes $2x + y - 3z = 2$ and $-x + 2y - z = 1$.

Solution:

We need to find a vector perpendicular to both normal vectors $\mathbf{n} = \langle 2, 1, -3 \rangle$ and $\mathbf{N} = \langle -1, 2, -1 \rangle$.

We choose $\mathbf{v} = \mathbf{N} \times \mathbf{n}$. That is,

$$\mathbf{v} = \mathbf{N} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -1 \\ 2 & 1 & -3 \end{vmatrix} = (-6 + 1)\,\mathbf{i} - (3 + 2)\,\mathbf{j} + (-1 - 4)\,\mathbf{k}$$

Result: $\mathbf{v} = \langle -5, -5, -5 \rangle$. A simpler choice is $\mathbf{v} = \langle 1, 1, 1 \rangle$. $\triangleleft$

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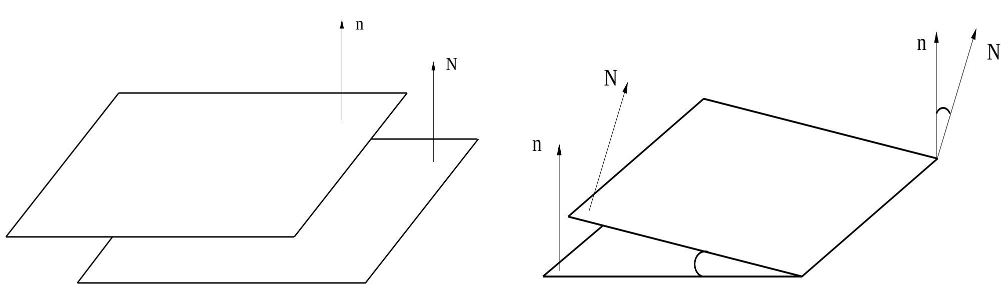
Parallel planes and angle between planes

Definition

Two planes are *parallel* if their normal vectors are parallel. The *angle* between two non-parallel planes is the smaller angle between their normal vectors.



Parallel planes and angle between planes

Example

Find the angle between the planes $2x + y - 3z = 2$ and $-x + 2y - z = 1$.

Solution: We need to find the angle between the normal vectors $\mathbf{n} = \langle 2, 1, -3 \rangle$ and $\mathbf{N} = \langle -1, 2, -1 \rangle$.

We use the dot product: $\cos(\theta) = \dfrac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}|\,|\mathbf{N}|}$.

The numbers we need are:

$$\mathbf{n} \cdot \mathbf{N} = -2 + 2 + 3 = 3,$$
$$|\mathbf{n}| = \sqrt{4 + 1 + 9} = \sqrt{14}, \quad |\mathbf{N}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

Therefore, $\cos(\theta) = 3/\sqrt{84}$. We conclude that

$$\theta = 70^\circ\ 53'\ 36''.$$

$\triangleleft$

Lines and planes in space (Sect. 12.5)

Planes in space.

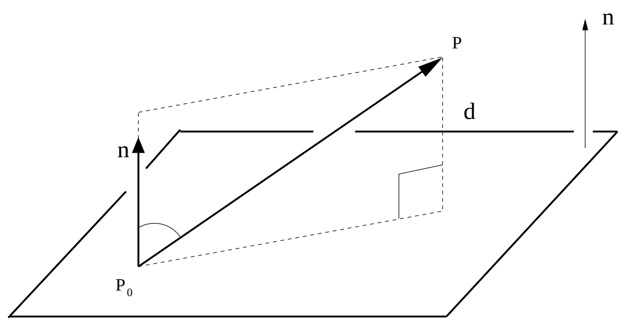
- Equations of planes in space.
  - Vector equation.
  - Components equation.
- The line of intersection of two planes.
- Parallel planes and angle between planes.
- **Distance from a point to a plane.**

Distance formula from a point to a plane

Theorem

The distance d from a point P to a plane containing P_0 with normal vector $\mathbf{n}$ is the shortest distance from P to any point in the plane, and is given by the expression

$$d = \frac{|(\overrightarrow{P_0P}) \cdot \mathbf{n}|}{|\mathbf{n}|}.$$

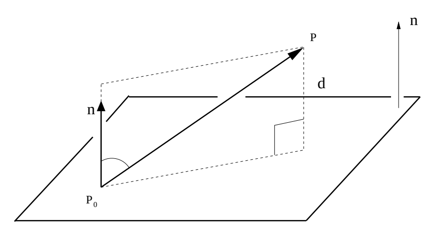


Distance formula from a point to a plane

Proof.

We need to proof the distance formula

$$d = \frac{|(\overrightarrow{P_0P}) \cdot \mathbf{n}|}{|\mathbf{n}|}.$$



From the picture we see that

$$d = \left| |\overrightarrow{P_0P}| \cos(\theta) \right|,$$

where θ is the angle between $\overrightarrow{P_0P}$ and $\mathbf{n}$, where the absolute value are needed since the distance is a non-negative number. Recall:

$$(\overrightarrow{P_0P}) \cdot \mathbf{n} = |\overrightarrow{P_0P}|\,|\mathbf{n}| \cos(\theta) \quad \Rightarrow \quad |\overrightarrow{P_0P}| \cos(\theta) = \frac{(\overrightarrow{P_0P}) \cdot \mathbf{n}}{|\mathbf{n}|}.$$

Take the absolute value above, and that is the formula for d. $\square$

Distance formula from a point to a plane

Example

Find the distance from the point $P = (1, 2, 3)$ to the plane $x - 3y + 2z = 4$.

Solution: We need to find a point P_0 on the plane and its normal vector $\mathbf{n}$. Then use the formula $d = |(\overrightarrow{P_0 P}) \cdot \mathbf{n}|/|\mathbf{n}|$.
A point on the plane is simple to find: Choose a point that intersects one of the axis, for example $y = 0$, $z = 0$, and $x = 4$.
That is, $P_0 = (4, 0, 0)$.
The normal vector is in the plane equation: $\mathbf{n} = \langle 1, -3, 2 \rangle$.
We now compute $\overrightarrow{P_0 P} = \langle -3, 2, 3 \rangle$. Then,

$$d = \frac{|-3-6+6|}{\sqrt{1+9+4}} \quad \Rightarrow \quad d = \frac{3}{\sqrt{14}}.$$

$\lhd$

Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- Quadratic surfaces:
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Cones, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$
  - Hyperboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$ $\quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Paraboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0.$
  - Saddles, $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0.$

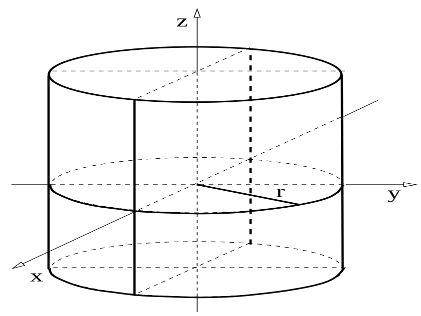
Cylinders.

Definition

Given a curve on a plane, called the *generating curve*, a *cylinder* is a surface in space generating by moving along the generating curve a straight line perpendicular to the plane containing the generating curve.

Example

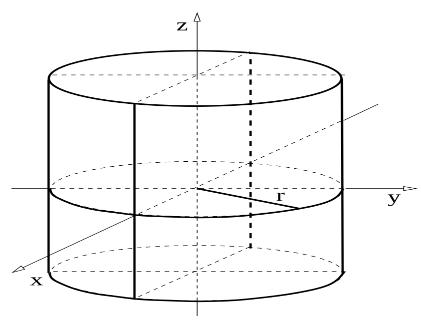
A *circular cylinder* is the particular case when the generating curve is a circle. In the picture, the generating curve lies on the xy-plane.



$\triangleleft$

Example

Find the equation of the cylinder given in the picture.



Solution:

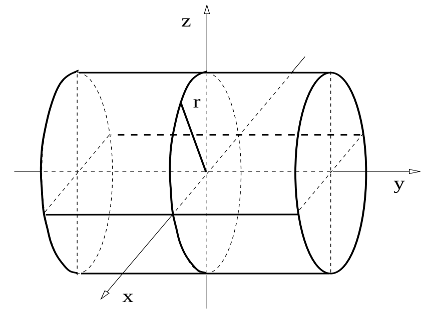
The intersection of the cylinder with the $z = 0$ plane is a circle with radius r, hence points of the form $(x, y, 0)$ belong to the cylinder iff $x^2 + y^2 = r^2$ and $z = 0$.

For $z \neq 0$, the intersection of horizontal planes of constant z with the cylinder again are circles of radius r, hence points of the form (x, y, z) belong to the cylinder iff $x^2 + y^2 = r^2$ and z constant.

Summarizing, the equation of the cylinder is $x^2 + y^2 = r^2$. We do not mention the coordinate z, since the equation above holds for every value of $z \in R$. $\triangleleft$

Example

Find the equation of the cylinder given in the picture.



Solution:

The generating curve is a circle, but this time on the plane $y = 0$. Hence point of the form $(x, 0, z)$ belong to the cylinder iff $x^2 + z^2 = r^2$.

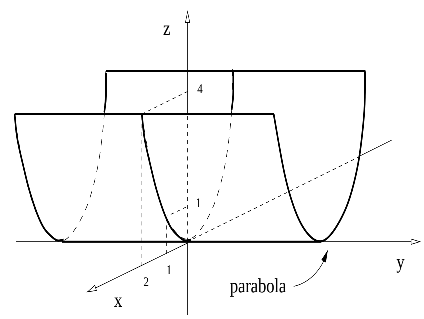
We conclude that the equation of the cylinder above is

$$x^2 + z^2 = r^2.$$

We do not mention the coordinate y, since the equation above holds for every value of $y \in \mathbb{R}$. $\triangleleft$

Example

Find the equation of the cylinder given in the picture.



Solution:

The generating curve is a parabola on planes with constant y.

This parabola contains the points $(0, 0, 0)$, $(1, 0, 1)$, and $(2, 0, 4)$.

Since three points determine a unique parabola and $z = x^2$ contains these points, then at $y = 0$ the generating curve is $z = x^2$.

The cylinder equation does not contain the coordinate y. Hence,

$$z = x^2, \qquad y \in \mathbb{R}.$$

$\triangleleft$

Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- **Quadratic surfaces:**
  - Spheres.
  - Ellipsoids.
  - Cones.
  - Hyperboloids.
  - Paraboloids.
  - Saddles.

Quadratic surfaces.

Definition

Given constants a_i, b_i and c_1, with $i = 1, 2, 3$, a *quadratic surface* in space is the set of points (x, y, z) solutions of the equation

$$a_1\, x^2 + a_2\, y^2 + a_3\, z^2 + b_1\, x + b_2\, y + b_3\, z + c_1 = 0.$$

Remark:

- The coefficients b_1, b_2, b_3 play a role moving around the surface in space.
- We study only quadratic equations of the form:

$$a_1\, x^2 + a_2\, y^2 + a_3\, z^2 + b_3\, z = c_2. \tag{1}$$

- The surfaces below are rotations of the one in Eq. (1),

$$a_1\, z^2 + a_2\, x^2 + a_3\, y^2 + b_3\, y = c_2,$$
$$a_1\, y^2 + a_2\, x^2 + a_3\, x^2 + b_3\, x = c_2.$$

Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- Quadratic surfaces:
  - **Spheres.** $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - Ellipsoids.
  - Cones.
  - Hyperboloids.
  - Paraboloids.
  - Saddles.

Spheres.

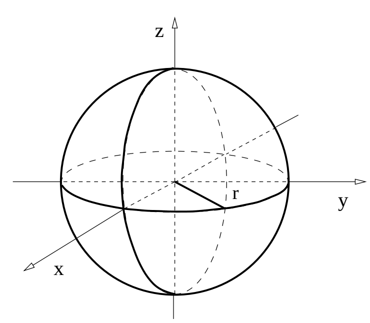
Recall: We study only quadratic equations of the form:

$$a_1\,x^2 + a_2\,y^2 + a_3\,z^2 + b_3\,z = c_2.$$

Example

A *sphere* is a simple quadratic surface, the one in the picture has the equation

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$$



($a_1 = a_2 = a_3 = 1/r^2$, $b_3 = 0$ and $c_2 = 1$.)
Equivalently, $x^2 + y^2 + z^2 = r^2$.

◁

Spheres.

Recall: Linear terms move the surface around in space.

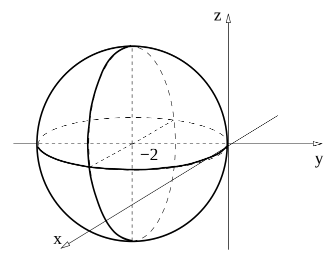
Example

Graph the surface given by the equation $x^2 + y^2 + z^2 + 4y = 0$.

Solution: Complete the square:

$$x^2 + \left[y^2 + 2\left(\frac{4}{2}\right)y + \left(\frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 + z^2 = 0.$$

Therefore, $x^2 + \left(y + \frac{4}{2}\right)^2 + z^2 = 4$. This is the equation of a sphere centered at $P_0 = (0, -2, 0)$ and with radius $r = 2$. $\triangleleft$



Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- **Quadratic surfaces:**
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - **Ellipsoids**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Paraboloids.
  - Cones.
  - Hyperboloids.
  - Saddles.

Ellipsoids.

Definition

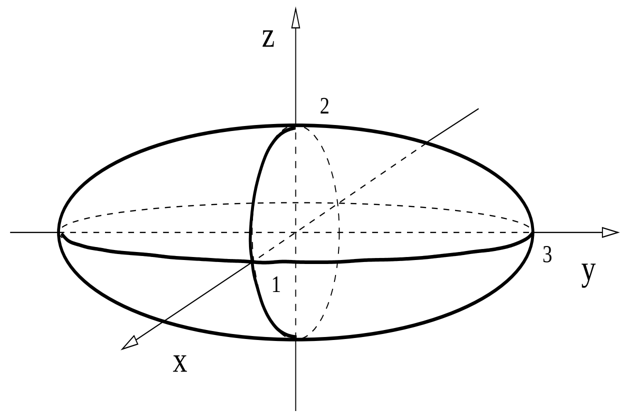
Given positive constants a, b, c, an *ellipsoid* centered at the origin is the set of point solution to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Example

Graph the ellipsoid,

$x^2 + \dfrac{y^2}{3^2} + \dfrac{z^2}{2^2} = 1.$ $\triangleleft$

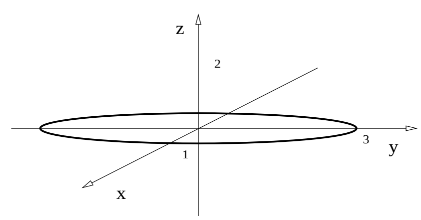


Example

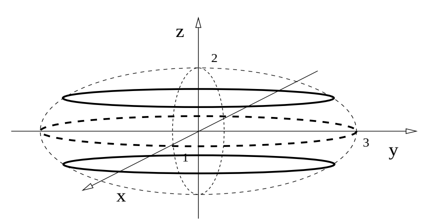
Graph the ellipsoid, $x^2 + \dfrac{y^2}{3^2} + \dfrac{z^2}{2^2} = 1.$

Solution:

On the plane $z = 0$ we have the ellipse $x^2 + \dfrac{y^2}{3^2} = 1.$



On the plane $z = z_0$, with $-2 < z_0 < 2$ we have the ellipse $x^2 + \frac{y^2}{3^2} = \left(1 - \frac{z_0^2}{2^2}\right)$. Denoting $c = 1 - (z_0^2/4)$, then $0 < c < 1$, and $\dfrac{x^2}{c} + \dfrac{y^2}{3^2 c} = 1.$ $\triangleleft$



Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- **Quadratic surfaces:**
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - **Cones**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$
  - Hyperboloids.
  - Paraboloids.
  - Saddles.

Cones.

Definition

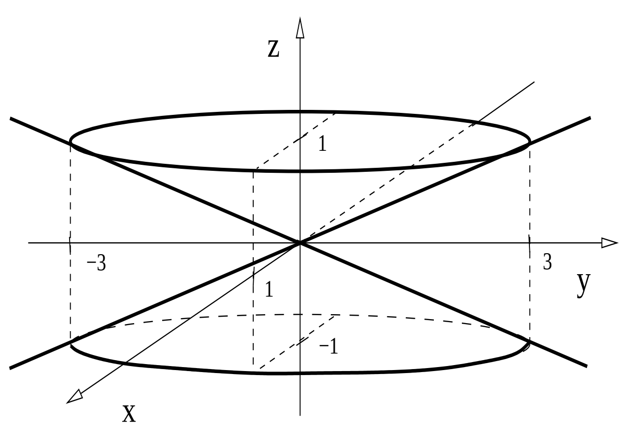
Given positive constants a, b, a *cone* centered at the origin is the set of point solution to the equation

$$z = \pm\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}.$$

Example

Graph the cone,

$z = \sqrt{x^2 + \frac{y^2}{3^2}}.$ ◁

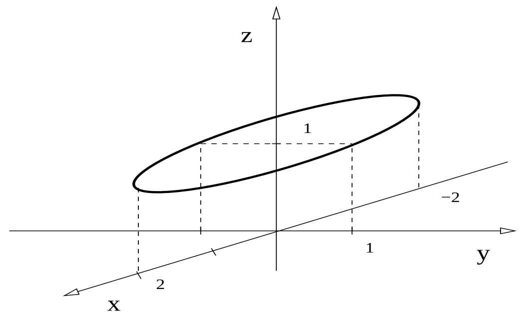


Example

Graph the cone, $z = +\sqrt{\frac{x^2}{2^2} + y^2}$.

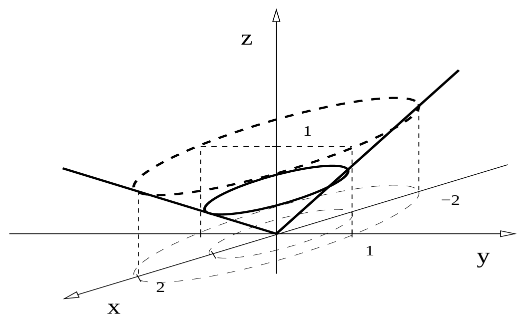
Solution:

On the plane $z = 1$ we have the ellipse $\dfrac{x^2}{2^2} + y^2 = 1$.



On the plane $z = z_0 > 0$ we have the ellipse $\dfrac{x^2}{2^2} + y^2 = z_0^2$, that is,

$$\frac{x^2}{2^2 z_0^2} + \frac{y^2}{z_0^2} = 1. \quad \triangleleft$$



Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- **Quadratic surfaces:**
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Cones, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$
  - **Hyperboloids**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Paraboloids.
  - Saddles.

Hyperboloids.

Definition

Given positive constants a, b, c, a *one sheet hyperboloid* centered at the origin is the set of point solution to the equation

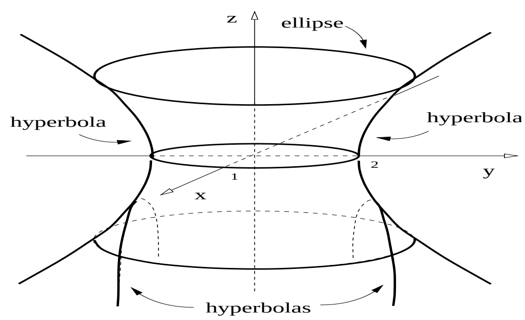
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(One negative sign, one sheet.)

Example

Graph the hyperboloid,

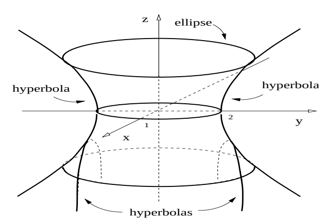
$x^2 + \dfrac{y^2}{2^2} - z^2 = 1.$ ◁



Hyperboloids.

Example

Graph the hyperboloid $\quad x^2 + \dfrac{y^2}{2^2} - z^2 = 1.$



Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On horizontal planes, $z = z_0$, we obtain ellipses
  $x^2 + \dfrac{y^2}{2^2} = 1 + z_0^2.$
- On vertical planes, $y = y_0$, we obtain hyperbolas
  $x^2 - z^2 = 1 - \dfrac{y_0^2}{2^2}.$
- On vertical planes, $x = x_0$, we obtain hyperbolas
  $\dfrac{y^2}{2^2} - z^2 = 1 - x_0^2.$

Hyperboloids.

Definition

Given positive constants a, b, c, a *two sheet hyperboloid* centered at the origin is the set of point solution to the equation

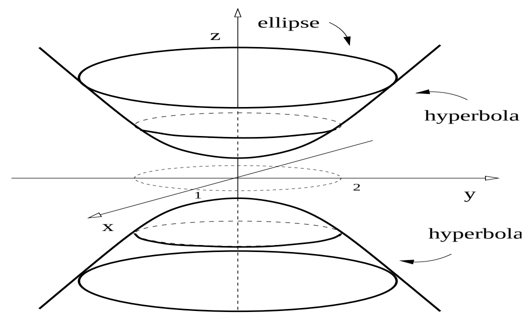
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(Two negative signs, two sheets.)

Example

Graph the hyperboloid,

$-x^2 - \frac{y^2}{2^2} + z^2 = 1.$ $\triangleleft$

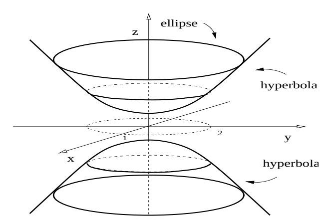


Hyperboloids.

Example

Graph the hyperboloid

$-x^2 - \frac{y^2}{2^2} + z^2 = 1.$



Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On horizontal planes, $z = z_0$, with $|z_0| > 1$, we obtain ellipses $x^2 + \frac{y^2}{2^2} = -1 + z_0^2.$
- On vertical planes, $y = y_0$, we obtain hyperbolas $-x^2 + z^2 = 1 + \frac{y_0^2}{2^2}.$
- On vertical planes, $x = x_0$, we obtain hyperbolas $-\frac{y^2}{2^2} + z^2 = 1 + x_0^2.$

Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- Quadratic surfaces:
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Cones, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$
  - Hyperboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$ $\quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - **Paraboloids**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0.$
  - Saddles.

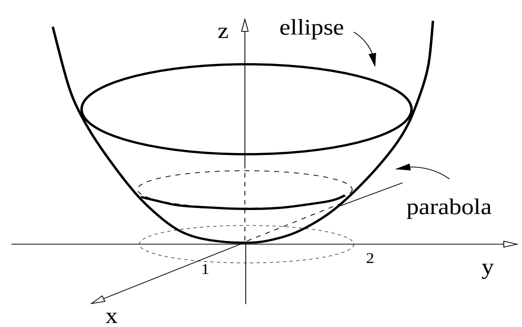
Paraboloids.

Definition
Given positive constants a, b, a *paraboloid* centered at the origin is the set of point solution to the equation

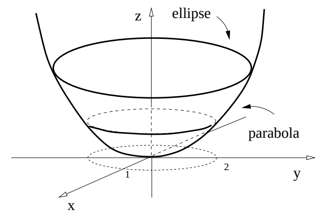
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Example
Graph the paraboloid,

$z = x^2 + \dfrac{y^2}{2^2}.$ ◁



Paraboloids.



Example

Graph the paraboloid $z = x^2 + \dfrac{y^2}{2^2}$.

Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On horizontal planes, $z = z_0$, with $z_0 > 0$, we obtain ellipses $x^2 + \dfrac{y^2}{2^2} = z_0$.
- On vertical planes, $y = y_0$, we obtain parabolas $z = x^2 + \dfrac{y_0^2}{2^2}$.
- On vertical planes, $x = x_0$, we obtain parabolas $z = x_0^2 + \dfrac{y^2}{2^2}$.

Cylinders and quadratic surfaces (Sect. 12.6).

- Cylinders.
- Quadratic surfaces:
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$.
  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
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  - Paraboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0$.
  - **Saddles**, $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0$.

Saddles, or hyperbolic paraboloids.

Definition

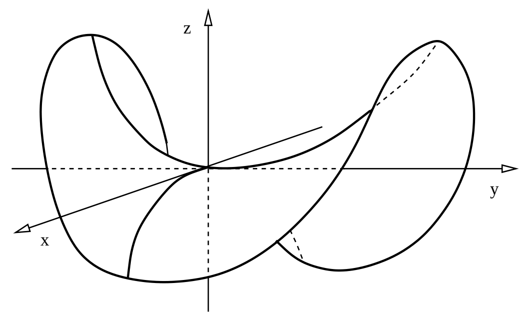
Given positive constants a, b, c, a *saddle* centered at the origin is the set of point solution to the equation

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Example

Graph the paraboloid,

$z = -x^2 + \dfrac{y^2}{2^2}$. $\triangleleft$

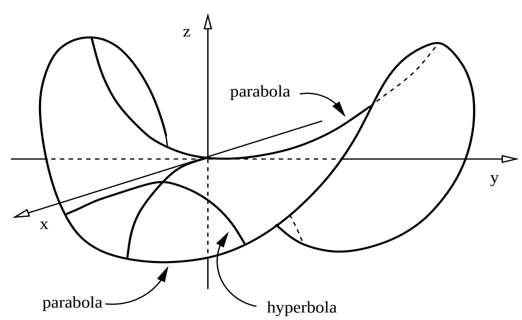


Saddles.

Example

Graph the saddle

$z = -x^2 + \dfrac{y^2}{2^2}$.



Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On planes, $z = z_0$, we obtain hyperbolas $-x^2 + \dfrac{y^2}{2^2} = z_0$.
- On planes, $y = y_0$, we obtain parabolas $z = -x^2 + \dfrac{y_0^2}{2^2}$.
- On planes, $x = x_0$, we obtain parabolas $z = -x_0^2 + \dfrac{y^2}{2^2}$.