Review for Exam 4.

- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) **Conservative fields, potential functions.**
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.
Conservative fields, potential functions (16.3).

Example
Is the field \( \mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle \) conservative? If “yes”, then find the potential function.

Solution: We need to check the equations

\[
\begin{align*}
\partial_y F_z &= \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x, \\
\partial_y F_z &= x \cos(z) = \partial_z F_y, \\
\partial_x F_z &= y \cos(z) = \partial_z F_x, \\
\partial_x F_y &= \sin(z) = \partial_y F_x.
\end{align*}
\]

Therefore, \( \mathbf{F} \) is a conservative field, that means there exists a scalar field \( f \) such that \( \mathbf{F} = \nabla f \). The equations for \( f \) are

\[
\begin{align*}
\partial_x f &= y \sin(z), \\
\partial_y f &= x \sin(z), \\
\partial_z f &= xy \cos(z).
\end{align*}
\]

Conservative fields, potential functions (16.3).

Example
Is the field \( \mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle \) conservative? If “yes”, then find the potential function.

Solution: \( \partial_x f = y \sin(z), \partial_y f = x \sin(z), \partial_z f = xy \cos(z) \).

Integrating in \( x \) the first equation we get

\[
f(x, y, z) = xy \sin(z) + g(y, z).
\]

Introduce this expression in the second equation above,

\[
\partial_y f = x \sin(z) + \partial_y g = x \sin(z) \quad \Rightarrow \quad \partial_y g(y, z) = 0,
\]

so \( g(y, z) = h(z) \). That is, \( f(x, y, z) = xy \sin(z) + h(z) \).

Introduce this expression into the last equation above,

\[
\partial_z f = xy \cos(z) + h'(z) = xy \cos(z) \quad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h(z) = c.
\]

We conclude that \( f(x, y, z) = xy \sin(z) + c \). \hspace{1cm} \triangleleft
Conservative fields, potential functions (16.3).

**Example**
Compute \( I = \int_C y \sin(z) \, dx + x \sin(z) \, dy + xy \cos(z) \, dz \), where \( C \) given by \( r(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle \) for \( t \in [0, 1] \).

**Solution:** We know that the field \( \mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle \) conservative, so there exists \( f \) such that \( \mathbf{F} = \nabla f \), or equivalently
\[
\text{df} = y \sin(z) \, dx + x \sin(z) \, dy + xy \cos(z) \, dz.
\]
We have computed \( f \) already, \( f = xy \sin(z) + c \).
Since \( \mathbf{F} \) is conservative, the integral \( I \) is path independent, and
\[
I = f(1, 2, \pi/2) - f(1, 1, \pi/2) = 2 \sin(\pi/2) - \sin(\pi/2) \Rightarrow I = 1.
\]

Conservative fields, potential functions (16.3).

**Example**
Show that the differential form in the integral below is exact,
\[
\int_C \left[ 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln(y) \, dz \right], \quad y > 0.
\]

**Solution:** We need to show that the field \( \mathbf{F} = \langle 3x^2, \frac{z^2}{y}, 2z \ln(y) \rangle \) is conservative. It is, since,
\[
\partial_y F_z = \frac{2z}{y} = \partial_z F_y, \quad \partial_x F_z = 0 = \partial_z F_x, \quad \partial_x F_y = 0 = \partial_y F_x.
\]
Therefore, exists a scalar field \( f \) such that \( \mathbf{F} = \nabla f \), or equivalently,
\[
df = 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln(y) \, dz.
\]
Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- (16.4) **The Green Theorem in a plane.**
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

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**The Green Theorem in a plane (16.4).**

**Example**

Use the Green Theorem in the plane to evaluate the line integral given by \( \int_C [(6y + x) \, dx + (y + 2x) \, dy] \) on the circle \( C \) defined by \((x - 1)^2 + (y - 3)^2 = 4\).

**Solution:** Recall: \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy \).

Here \( \mathbf{F} = \langle (6y + x), (y + 2x) \rangle \). Since \( \partial_x F_y = 2 \) and \( \partial_y F_x = 6 \), Green’s Theorem implies

\[
\oint_C [(6y + x) \, dx + (y + 2x) \, dy] = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (2 - 6) \, dx \, dy.
\]

Since the area of the disk \( S = \{(x - 1)^2 + (y - 3)^2 \leq 4\} \) is \( \pi(2^2) \),

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = -4 \iint_S dx \, dy = -4(4\pi) \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = -16\pi.
\]
Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) **Surface area, surface integrals.**
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

**Surface area, surface integrals (16.5).**

**Example**

Integrate the function \( g(x, y, z) = x\sqrt{4 + y^2} \) over the surface cut from the parabolic cylinder \( z = 4 - y^2/4 \) by the planes \( x = 0 \), \( x = 1 \) and \( z = 0 \).

**Solution:**

We must compute: \( I = \iint_S g \, d\sigma \).

Recall \( d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} \, dx \, dy \), with \( k \perp R \)

and in this case \( f(x, y, z) = y^2 + 4z - 16 \).

\[
\nabla f = \langle 0, 2y, 4 \rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}.
\]

Since \( R = [0, 1] \times [-4, 4] \), its normal vector is \( k \) and \( |\nabla f \cdot k| = 4 \).

Then,

\[
\iint_S g \, d\sigma = \iint_R \left(x\sqrt{4 + y^2}\right) \frac{2\sqrt{4 + y^2}}{4} \, dx \, dy.
\]
Surface area, surface integrals (16.5).

Example

Integrate the function $g(x, y, z) = x\sqrt{4 + y^2}$ over the surface cut from the parabolic cylinder $z = 4 - y^2 / 4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution:

\[
\iint_S g \, d\sigma = \iint_R \left( x\sqrt{4 + y^2} \right) \frac{2\sqrt{4 + y^2}}{4} \, dx \, dy.
\]

\[
\iint_S g \, d\sigma = \frac{1}{2} \iint_R x(4 + y^2) \, dx \, dy = \frac{1}{2} \int_{-4}^{4} \int_{0}^{1} x(4 + y^2) \, dx \, dy.
\]

\[
\iint_S g \, d\sigma = \frac{1}{2} \left[ \int_{-4}^{4} (4 + y^2) \, dy \right] \left[ \int_{0}^{1} x \, dx \right] = \frac{1}{2} \left[ 4y + \frac{y^3}{3} \right]_{-4}^{4} \left[ \frac{x^2}{2} \right]_{0}^{1}.
\]

\[
\iint_S g \, d\sigma = \frac{1}{2} \left( 4^2 + \frac{4^3}{3} \right) \frac{1}{2} = 8 \left( 1 + \frac{4}{3} \right) \quad \Rightarrow \quad \iint_S g \, d\sigma = \frac{56}{3}.
\]

Review for Exam 4.

- (16.1) Line integrals.
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- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- **(16.7) The Stokes Theorem.**
- (16.8) The Divergence Theorem.
Example
Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface $S$, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and $S = \{x^2 + y^2 = a^2, \ z \in [0, h] \} \cup \{x^2 + y^2 \leq a^2, \ z = h \}$.

Solution: Recall: $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

The surface $S$ is the cylinder walls and its cover at $z = h$.
Therefore, the curve $C$ is the circle $x^2 + y^2 = a^2$ at $z = 0$.
That circle can be parametrized (counterclockwise) as $\mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, 2\pi]$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt,$$
where $\mathbf{F}(t) = \langle -a \sin(t), a \cos(t), a^2 \cos^2(t) \rangle$ and $\mathbf{r}'(t) = \langle -a \sin(t), a \cos(t), 0 \rangle$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = 2\pi a^2.$$
Review for Exam 4.

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- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

The Divergence Theorem (16.8).

**Example**

Use the Divergence Theorem to find the outward flux of the field \( \mathbf{F} = \langle x^2, -2xy, 3xz \rangle \) across the boundary of the region 
\( D = \{ x^2 + y^2 + z^2 \leq 4, \ x \geq 0, \ y \geq 0, \ z \geq 0 \} \).

**Solution:** Recall: 
\[ \int \int _{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma \, = \, \int \int \int _{D} (\nabla \cdot \mathbf{F}) \, dv. \]

\[ \nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 3x. \]

\[ \int \int _{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int \int \int _{D} (\nabla \cdot \mathbf{F}) \, dv = \int \int \int _{D} 3x \, dx \, dy \, dz. \]

It is convenient to use spherical coordinates:

\[ \int \int _{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int _{0}^{\pi / 2} \int _{0}^{\pi / 2} \int _{0}^{2} [3\rho \sin(\phi) \cos(\phi)] \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta. \]
The Divergence Theorem (16.8).

Example
Use the Divergence Theorem to find the outward flux of the field \( \mathbf{F} = \langle x^2, -2xy, 3xz \rangle \) across the boundary of the region \( D = \{ x^2 + y^2 + z^2 \leq 4, \ x \geq 0, \ y \geq 0, \ z \geq 0 \} \).

Solution:
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 [3\rho \sin(\phi) \cos(\phi)] \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta.
\]
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \left[ \int_0^{\pi/2} \cos(\theta) \, d\theta \right] \left[ \int_0^{\pi/2} \sin^2(\phi) \, d\phi \right] \left[ \int_0^2 3\rho^3 \, d\rho \right]
\]
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \left[ \sin(\theta) \left|_0^{\pi/2} \right. \right] \frac{1}{2} \int_0^{\pi/2} (1 - \cos(2\phi)) \, d\phi \left[ \frac{3}{4} \rho^4 \right]_0^2
\]
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = (1) \frac{1}{2} \left( \frac{\pi}{2} \right) (12) \quad \Rightarrow \quad \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 3\pi.
\]

Review for the Final Exam.

- Monday, December 13, 10:00am - 12:00 noon. (2 hours.)
- Places:
  - Sctns 001, 002, 005, 006 in E-100 VMC (Vet. Medical Ctr.),
  - Sctns 003, 004, in 108 EBH (Ernst Bessey Hall);
  - Sctns 007, 008, in 339 CSE (Case Halls).
- Chapters 12-16.
- Problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.
Chapter 16, Integration in vector fields.

Example

Use the Divergence Theorem to find the flux of $F = (xy^2, x^2y, y)$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$, and $z = 1$.

Solution: Recall: $\int \int_S F \cdot n \, d\sigma = \int \int \int_D (\nabla \cdot F) \, dv$. We start with

$$\nabla \cdot F = \partial_x (xy^2) + \partial_y (x^2y) + \partial_z (y) \quad \Rightarrow \quad \nabla \cdot F = y^2 + x^2.$$ 

The integration region is $D = \{x^2 + y^2 \leq 1, \ z \in [-1, 1]\}$. So,

$$I = \int \int \int_D (\nabla \cdot F) \, dv = \int \int \int_D (x^2 + y^2) \, dx \, dy \, dz.$$ 

We use cylindrical coordinates,

$$I = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 \, dz \, r \, dr \, d\theta = 2\pi \left[ \int_0^1 r^3 \, dr \right] (2) = 4\pi \left( \frac{r^4}{4} \right) \bigg|_0^1.$$ 

We conclude that $\int \int_S F \cdot n \, d\sigma = \pi$. \hfill \triangle
Example

Use Stokes' Theorem to find the work done by the force $F = \langle 2xz, xy, yz \rangle$ along the path $C$ given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:

Recall: $\int_{C} F \cdot dr = \iint_{S} (\nabla \times F) \cdot n \, d\sigma$.

The surface $S$ is the level surface $f = 0$ of $f = x + y + z - 1$.

Therefore, $\nabla f = \langle 1, 1, 1 \rangle$, $|\nabla f| = \sqrt{3}$, and $|\nabla f \cdot k| = 1$.

$n = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$, $d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} \, dx \, dy = \sqrt{3} \, dx \, dy$.

We now compute the curl of $F$,

$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_{x} & \partial_{y} & \partial_{z} \\ 2xz & xy & yz \end{vmatrix} = \langle (z - 0), -(0 - 2x), (y - 0) \rangle$.

So $\nabla \times F = \langle z, 2x, y \rangle$. Therefore,

$\iint_{S} (\nabla \times F) \cdot n \, d\sigma = \iint_{R} \left( \langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) \sqrt{3} \, dx \, dy$
Chapter 16, Integration in vector fields.

Example
Use Stokes’ Theorem to find the work done by the force \( \mathbf{F} = \langle 2xz, xy, yz \rangle \) along the path \( C \) given by the intersection of the plane \( x + y + z = 1 \) with the first octant, counterclockwise when viewed from above.

Solution:
\[
I = \iiint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \iint_R \left( \langle z, 2x, y \rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle \right) \sqrt{3} \, dx \, dy.
\]
\[
I = \iint_R (z + 2x + y) \, dx \, dy,
\]
\[
I = x \bigg|_0^1 \frac{1}{3} x^3 \bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3}.
\]
Chapter 16, Integration in vector fields.

Example
Find the area of the cone \( S \) given by \( z = \sqrt{x^2 + y^2} \) for \( z \in [0, 1] \).
Also find the flux of the field \( \mathbf{F} = \langle x, y, 0 \rangle \) outward through \( S \).

Solution: We now compute the outward flux \( I = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \).
Since
\[
\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}z} \langle x, y, -z \rangle.
\]
\[
I = \iiint_R \frac{1}{\sqrt{2}z} (x^2 + y^2) \sqrt{2} \, dx \, dy = \iint_R \sqrt{x^2 + y^2} \, dx \, dy.
\]
Using polar coordinates, we obtain
\[
I = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \frac{2\pi}{3} \left|_0^1 \right. \Rightarrow \quad I = \frac{2\pi}{3}.
\]

Review for Final Exam.

- **Chapter 15, Sections 15.1-15.4, 15.6.**
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1, 13.3.
- Chapter 12, Sections 12.1-12.6.
Example

Find the volume of the region bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:

So, $D = \{x^2 + y^2 \leq 1, \ 0 \leq z \leq 1 - x^2 - y^2\}$, and $R = \{x^2 + y^2 \leq 1, \ z = 0\}$. We know that

$$V(D) = \iiint_{D} \, dv = \iiint_{R} \int_{0}^{1-x^2-y^2} \, dz \, dx \, dy.$$  

Using cylindrical coordinates $(r, \theta, z)$, we get

$$V(D) = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{1-r^2} \, dz \, dr \, d\theta = 2\pi \int_{0}^{1} (1 - r^2) \, r \, dr.$$  

Substituting $u = 1 - r^2$, so $du = -2r \, dr$, we obtain

$$V(D) = 2\pi \int_{1}^{0} u \left(-\frac{du}{2}\right) = \pi \int_{0}^{1} u \, du = \pi \left[ \frac{u^2}{2} \right]_{0}^{1} \Rightarrow \ V(D) = \frac{\pi}{2}.$$  

Example

Set up the integrals needed to compute the average of the function $f(x, y, z) = z \sin(x)$ on the bounded region $D$ in the first octant bounded by the plane $z = 4 - 2x - y$. Do not evaluate the integrals.

Solution: Recall: $\bar{f} = \frac{1}{V(D)} \iiint_{D} \, f \, dv$.

Since $V(D) = \int_{0}^{2} \int_{0}^{4-2x} \int_{0}^{4-2x-y} \, dz \, dy \, dx,$

we conclude that

$$\bar{f} = \int_{0}^{2} \int_{0}^{4-2x} \int_{0}^{4-2x-y} \frac{z \sin(x)}{dz \, dy \, dx}.$$  

$$\bar{f} = \int_{0}^{2} \int_{0}^{4-2x} \int_{0}^{4-2x-y} \frac{z \sin(x)}{dz \, dy \, dx}.$$  

$$\bar{f} = \int_{0}^{2} \int_{0}^{4-2x} \int_{0}^{4-2x-y} \frac{z \sin(x)}{dz \, dy \, dx}.$$  

$$\bar{f} = \int_{0}^{2} \int_{0}^{4-2x} \int_{0}^{4-2x-y} \frac{z \sin(x)}{dz \, dy \, dx}.$$
Example
Reverse the order of integration and evaluate the double integral
\[ I = \int_0^4 \int_{y/2}^2 e^{x^2} \, dx \, dy. \]

Solution: We see that \( y \in [0, 4] \) and \( x \in [0, y/2] \), that is,

Therefore, reversing the integration order means
\[ I = \int_0^2 \int_0^{2x} e^{x^2} \, dy \, dx. \]

This integral is simple to compute,
\[ I = \int_0^2 e^{x^2} \, dx, \quad u = x^2, \quad du = 2x \, dx, \]
\[ I = \int_0^4 e^u \, du \Rightarrow I = e^4 - 1. \]

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- Chapters 12-16.
- \(~12 \) Problems, similar to homework problems.
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Remark on Chapter 16.

Remark: The normal form of Green’s Theorem is a two-dimensional restriction of the Divergence Theorem.

- The Divergence Theorem: \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) \, dv \).
- Normal form of Green’s Thrm: \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S (\nabla \cdot \mathbf{F}) \, dA \).

Remark: The tangential form of Green’s Theorem is a particular case of the Stokes Theorem when \( C, S \) are flat (on \( z = 0 \) plane).

- The Stokes Theorem: \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iiint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma \).
- Tang. form of Green’s Thrm: \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA \).


Example

Given \( A = (1, 2, 3), B = (6, 5, 4) \) and \( C = (8, 9, 7) \), find the following:

- \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \).
  Solution: \( \overrightarrow{AB} = \langle 6 - 1, 5 - 2, 4 - 3 \rangle \), hence \( \overrightarrow{AB} = \langle 5, 3, 1 \rangle \). In the same way \( \overrightarrow{AC} = \langle 7, 7, 4 \rangle \).
- \( \overrightarrow{AB} + \overrightarrow{AC} = \langle 12, 10, 5 \rangle \).
- \( \overrightarrow{AB} \cdot \overrightarrow{AC} = 35 + 21 + 4 \).
- \( \overrightarrow{AB} \times \overrightarrow{AC} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 1 \\ 7 & 7 & 4 \end{array} \right| = \langle 12 - 7, -(20 - 7), (35 - 21) \rangle \), hence \( \overrightarrow{AB} \times \overrightarrow{AC} = \langle 5, -13, 14 \rangle \).
Example

Find the parametric equation of the line through the point 
\((1, 0, -1)\) and perpendicular to the plane \(2x - 3y + 5x = 35\). Then find the intersection of the line and the plane.

Solution: The normal vector to the plane \(\langle 2, -3, 5 \rangle\) is the tangent vector to the line. Therefore,

\[ \mathbf{r}(t) = \langle 1, 0, -1 \rangle + t \langle 2, -3, 5 \rangle, \]

so the parametric equations of the line are

\[ x(t) = 1 + 2t, \quad y(t) = -3t, \quad z(t) = -1 + 5t. \]

The intersection point has a \(t\) solution of

\[ 2(1+2t)-3(-3t)+5(-1+5t) = 35 \quad \Rightarrow \quad 2+4t+9t-5+25t = 35 \]

\[ 38t = 38 \quad \Rightarrow \quad t = 1 \quad \Rightarrow \quad \mathbf{r}(1) = \langle 3, -3, 4 \rangle. \]

Example

The velocity of a particle is given by \( \mathbf{v}(t) = \langle t^2, (t^3 + 1) \rangle \), and the particle is at \( \langle 2, 1 \rangle \) for \( t = 0 \).

- Where is the particle at \( t = 2 \)?

Solution: \( \mathbf{r}(t) = \langle \left( \frac{t^3}{3} + r_x \right), \left( \frac{t^4}{4} + t + r_y \right) \rangle \). Since \( \mathbf{r}(0) = \langle 2, 1 \rangle \), we get that \( \mathbf{r}(t) = \langle \left( \frac{t^3}{3} + 2 \right), \left( \frac{t^4}{4} + t + 1 \right) \rangle \).

Hence \( \mathbf{r}(2) = \langle 8/3 + 2, 7 \rangle \).

- Find an expression for the particle arc length for \( t \in [0, 2] \).

Solution: \( s(t) = \int_0^t \sqrt{r^4 + (r^3 + 1)^2} \, dr \).

- Find the particle acceleration.

Solution: \( \mathbf{a}(t) = \langle 2t, 3t^2 \rangle \).

Example

- Draw a rough sketch of the surface $z = 2x^2 + 3y^2 + 5$.

  Solution: This is a paraboloid along the vertical direction, opens up, with vertex at $z = 5$ on the $z$-axis, and the $x$-radius is a bit longer than the $y$-radius.

- Find the equation of the tangent plane to the surface at the point $(1, 1, 10)$.

  Solution: Introduce $f(x, y) = 2x^2 + 3y^2 + 5$, then

  $$L_{(1,1)}(x, y) = \partial_x f(1, 1)(x - 1) + \partial_y f(1, 1)(y - 1) + f(1, 1).$$

  Since $f(1, 1) = 10$, and $\partial_x f = 4x$, $\partial_y f = 6y$, then

  $$z = L_{(1,1)}(x, y) = 4(x - 1) + 6(y - 1) + 10.$$


Example

Let $w = f(x, y)$ and $x = s^2 + t^2$, $y = st^2$. If $\partial_x f = x - y$ and $\partial_y f = y - x$, find $\partial_s w$ and $\partial_t w$ in terms of $s$ and $t$.

Solution:

$$\partial_s w = \partial_x f \partial_s x + \partial_y f \partial_s y = (x - y)2s + (y - x)t^2 = (x - y)(2s - t^2).$$

Therefore, $\partial_s w = (s^2 + t^2 - st^2)(2s - t^2)$.

$$\partial_t w = \partial_x f \partial_t x + \partial_y f \partial_t y = (x - y)2t + (y - x)2st = (x - y)(2t - 2st).$$

Therefore, $\partial_t w = (s^2 + t^2 - st^2)2t(1 - s)$.
Example
Find all critical points of the function $f(x, y) = 2x^2 + 8xy + y^4$ and determine whether they re local maximum, minimum of saddle points.

Solution:

\[ \nabla f = ((4x + 8y), (8x + 4y^3)) = (0, 0) \quad \Rightarrow \quad \begin{cases} x + 2y = 0, \\ 2x + y^3 = 0. \end{cases} \]

\[ -4y + y^3 = 0 \quad \Rightarrow \quad \begin{cases} y = 0 \Rightarrow x = 0 \Rightarrow P_0 = (0, 0) \\ y = \pm 2 \Rightarrow x = \mp 4 \Rightarrow P_1 = (4, -2), P_2 = (-4, 2) \end{cases} \]

Since $f_{xx} = 4$, $f_{yy} = 12y^2$, and $f_{xy} = 8$, we conclude that $D = 3(16)y^2 - 4(16)$.


Example
Find all critical points of the function $f(x, y) = 2x^2 + 8xy + y^4$ and determine whether they re local maximum, minimum of saddle points.

Solution:

$P_0 = (0, 0), P_1 = (4, -2), P_2 = (-4, 2), D = 3(16)y^2 - 4(16)$.

$D(0, 0) = -4(16) < 0 \quad \Rightarrow \quad P_0 = (0, 0) \text{ saddle point.}$

$D(4, -2) = 12(16) - 4(16) > 0, \quad f_{xx} = 4 \Rightarrow P_1 = (4, -2) \text{ min.}$

$D(-4, 2) = 12(16) - 4(16) > 0, \quad f_{xx} = 4 \Rightarrow P_1 = (-4, 2) \text{ min.}$
**Example**

Evaluate the integral \( I = \int_0^1 \int_x^\sqrt{x} y \, dy \, dx \) by reversing the order of integration.

**Solution:** The integration region is the set in the square \([0, 1] \times [0, 1]\) in between the curves \( y = x \) and \( y = \sqrt{x} \). Therefore,
\[
I = \int_0^1 \int_y^y y \, dx \, dy = \int_0^1 y(y - y^2) \, dy = \int_0^1 (y^2 - y^3) \, dy
\]
\[
I = \left[ \frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} \quad \Rightarrow \quad I = \frac{1}{12}.
\]

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**Practice final exam: April 30, 2001. Prbl. 8.**

**Example**

Find the work done by the force \( \mathbf{F} = \langle yz, xz, -xy \rangle \) on a particle moving along the path \( \mathbf{r}(t) = \langle t^3, t^2, t \rangle \) for \( t \in [0, 2] \).

**Solution:**
\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt,
\]
where \( \mathbf{F}(t) = \langle t^3, t^4, -t^5 \rangle \) and \( \mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle \). Hence
\[
W = \int_0^2 (3t^5 + 2t^5 - t^5) \, dt = \int_0^2 4t^5 \, dt = \frac{4}{6} t^6 \bigg|_0^2 = \frac{2}{3} 2^6.
\]
Therefore, \( W = 2^7 / 3 \).
Example
Show that the force field
\[ \mathbf{F} = \langle (y \cos(z) - yze^x), (x \cos(z) - ze^x), (-xy \sin(z) - ye^x) \rangle \]
is conservative. Then find its potential function. Then evaluate
\[ I = \int_C \mathbf{F} \cdot d\mathbf{r} \text{ for } \mathbf{r}(t) = \langle t, t^2, \pi t^3 \rangle. \]

Solution: The field \( \mathbf{F} \) is conservative, since
\[ \partial_x F_y = \cos(z) - ze^x = \partial_y F_x, \]
\[ \partial_x F_z = -xy \sin(z) - ye^x = \partial_z F_x, \]
\[ \partial_y F_z = -x \sin(z) - e^x = \partial_z F_y. \]
The potential function is a scalar function \( f \) solution of
\[ \partial_x f = y \cos(z) - yze^x, \quad \partial_y f = x \cos(z) - ze^x, \quad \partial_z f = -xy \sin(z) - ye^x. \]
Example
Show that the force field
\[ F = \langle (y \cos(z) - yze^x), (x \cos(z) - ze^x), (-xy \sin(z) - ye^x) \rangle \]
is conservative. Then find its potential function. Then evaluate
\[ I = \int_C F \cdot dr \] for \( r(t) = \langle t, t^2, \pi t^3 \rangle \).

Solution: Recall: \( f = xy \cos(z) - yze^x + h(z) \).
Introduce \( f \) into the equation \( \partial_z f = -xy \sin(z) - ye^x \), that is,
\[ -xy \sin(z) - e^x + h'(z) = -xy \sin(z) - ye^x \quad \Rightarrow \quad h'(z) = 0. \]
So, \( h(z) = c \), a constant, hence \( f = xy \cos(z) - yze^x + c \).
Finally \[ \int_C F \cdot dr = \int_{(0,0,0)}^{(1,1,\pi)} df = f(1,1,\pi) - f(0,0,0). \]
So we conclude that \[ \int_C F \cdot dr = -(1 + \pi e). \]

A straightforward calculation gives \[ \int_C F \cdot dr = 3. \]
Example
Find the surface area of the portion of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the plane \( z = 0 \). Use polar coordinates to evaluate the integral.

Solution:
\[
A(S) = \int \int_S d\sigma, \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} \, dx \, dy
\]
where \( f = x^2 + y^2 + z - 4 \). Therefore,
\[
\nabla f = \langle 2x, 2y, 1 \rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{1 + 4x^2 + 4y^2}, \quad \nabla f \cdot k = 1.
\]

\[
A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta, \quad u = 1 + 4r^2, \quad du = 8r \, dr.
\]
The finally obtain \( A(S) = (\pi/6)(17^{3/2} - 1) \).

Example
Use the Stokes Theorem to evaluate \( I = \int \int_S [\nabla \times (yi)] \cdot n \, d\sigma \)
where \( S \) is the hemisphere \( x^2 + y^2 + z^2 = 1 \), with \( z \geq 0 \).

Solution: \( F = \langle y, 0, 0 \rangle \). The border of the hemisphere is given by the circle \( x^2 + y^2 = 1 \), with \( z = 0 \). This circle can be parametrized for \( t \in [0, 2\pi] \) as
\[
r(t) = \langle \cos(t), \sin(t), 0 \rangle \quad \Rightarrow \quad r'(t) = \langle -\sin(t), \cos(t), 0 \rangle,
\]
and we also have \( F(t) = \langle \sin(t), 0, 0 \rangle \). Therefore,
\[
\int \int_S (\nabla \times F) \cdot n \, d\sigma = \int_0^{2\pi} F(t) \cdot r'(t) \, dt = -\int_0^{2\pi} \sin^2(t) \, dt
\]
\[
\int \int_S (\nabla \times F) \cdot n \, d\sigma = -\frac{1}{2} \int_0^{2\pi} [1 - \cos(2t)] \, dt.
\]
Example

Use the Stokes Theorem to evaluate $I = \iint_S \left[ \nabla \times (y\mathbf{i}) \right] \cdot n \, d\sigma$
where $S$ is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geq 0$.

Solution: $\iint_S (\nabla \times \mathbf{F}) \cdot n \, d\sigma = -\frac{1}{2} \int_0^{2\pi} \left[ 1 - \cos(2t) \right] \, dt$.
Recall that

$$\int_0^{2\pi} \cos(2t) \, dt = \frac{1}{2} \left( \sin(2t) \right)_{0}^{2\pi} = 0.$$

Therefore, we obtain

$$\iint_S (\nabla \times \mathbf{F}) \cdot n \, d\sigma = -\pi.$$